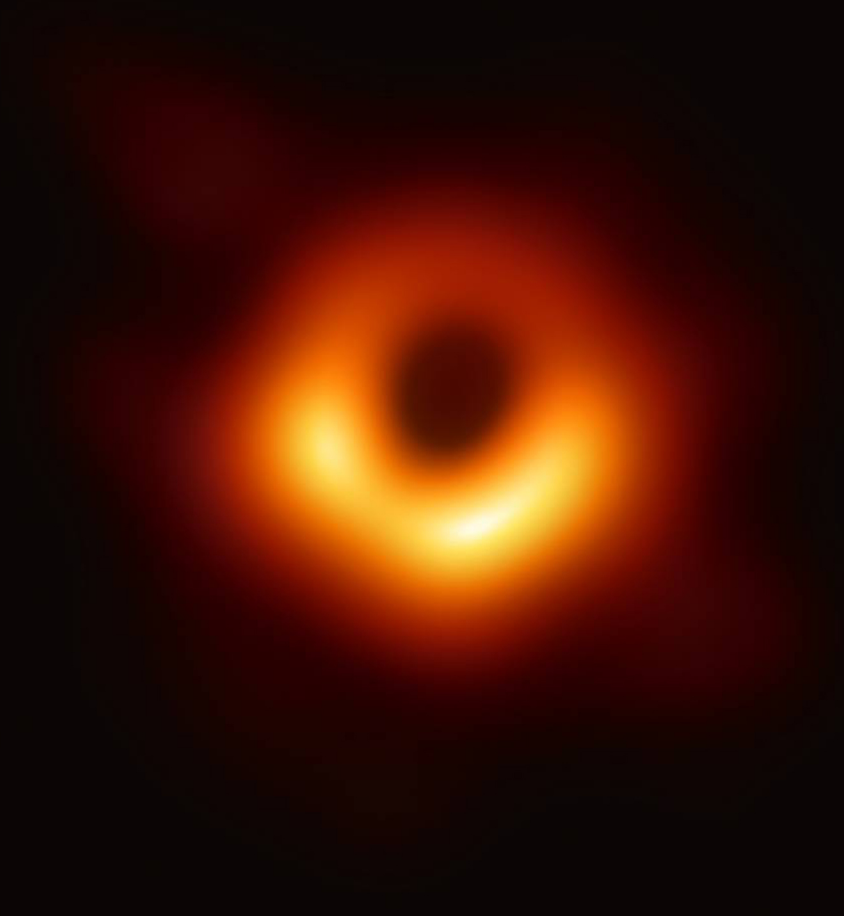
Holography in Out of Equilibrium Systems and Asymptotic Symmetries of Black Holes



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Cover image: Black hole at the center of galaxy M87. Event Horizon Telescope Collaboration (EHT).

Memoria de tesis doctoral

Holography in Out of Equilibrium Systems and Asymptotic Symmetries of Black Holes

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A mis padres

ABSTRACT

This thesis, presented in the form of a "compilation of articles", touches different areas of Theoretical Physics.

The first part, which comprises the first two articles, focuses on using applied holography to study out-of-equilibrium physics. More specifically, in the first article we study quantum processes called *quenches* and we show, by holographically constructing gravitational dual backgrounds, the way to implement a quantum quench with a real change in the hamiltonian in a Lorentz-invariant manner. The second article is framed under the theory of anomalous transport. The object of study is the gravitational contribution to the chiral anomaly, which through the chiral magnetic effect induces an energy current proportional to the square of the temperature when the system is in equilibrium. In this framework, we implement holographic quantum quenches to bring the system out of equilibrium, where no temperature can be well defined. Our results indicate a strong suppression of this effect when the system is very far from equilibrium.

The second part of the thesis lies in the context of exact, non-linear general relativity. In the third article, we characterize generic, non-expanding black hole horizons and study the effect of supertranslations on them. We present a freely specifiable data set which is both necessary and sufficient to reconstruct the full horizon geometry and is composed of objects that are invariant under supertranslations. Therefore we conclude that these transformations do not transform the geometry of the horizon and should be regarded as pure gauge.

RESUMEN

Esta tesis, presentada en la forma de "compilación de artículos", toca diferentes areas de la Física Teórica.

La primera parte, que consta de los dos primeros artículos, se centra en usar holografía aplicada al estudio de la física fuera del equilibrio. Más concretamente, en el primer artículo estudiamos procesos cuánticos llamados *quenches* y demostramos, construyendo holográficamente los escenarios gravitacionales de la teoría dual, la manera de implementar un quench cuántico con un cambio real en el hamiltoniano de manera invariante Lorentz. El segundo artículo se encuadra bajo la teoría de transporte anómalo. El objeto de estudio es la contribución gravitacional a la anomalía quiral, que a través del efecto magnético quiral induce una corriente de energía que es proporcional al cuadrado de la temperatura cuando el sistema está en equilibrio. En este escenario implementamos quenches holográficos para llevar al sistema fuera del equilibrio, donde la temperatura no está bien definida. Nuestros resultados indican una fuerte supresión de este efecto cuando el sistema está lejos de encontrarse en equilibrio.

La segunda parte de esta tesis se encuentra en el contexto de la relatividad general exacta y no-lineal. En el tercer artículo, caracterizamos horizontes genéricos y que no se expanden de agujeros negros, y estudiamos el efecto que tienen las supertraslaciones en ellos. Presentamos un conjunto de datos que pueden especificarse libremente, que son necesarios y suficientes para reconstruir la geometría completa del horizonte y que son invariante bajo supertraslaciones. Por lo tanto concluimos que estas transformaciones no transforman la geometría del horizonte y deben ser consideradas como puro gauge.

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CONTENTS

INTRODUCTION

The Standard Model of particle physics is a Quantum Field Theory (QFT) that describes extremely well the nature of the particles of our Universe at a microscopic level. However, it is far from being the ultimate theory of the Universe, and it has a long (and unfinished) list of open problems on which theoretical physicists work to understand. One main issue is that the Standard Model explains the weak, the strong and the electromagnetic interactions, but it does not take into account gravity. The lack of a consistent quantum theory of gravity is, at least to me, the greatest challenge in theoretical physics nowadays. Although String Theory is a strong candidate, it still has several loose ends, and it is not clear that it will succeed. Another open issue is strongly coupled physics. The strong interaction, Quantum Chromodynamics (QCD), unlike the other 3 forces of nature, is weakly coupled at high energies and strongly coupled at low energies. This leads to phenomena such as confinement or asymptotic freedom. The problem is that the framework used to actually compute things and make predictions in QFT, perturbation theory, is not valid for QCD at low energies, and new ideas such as lattice field theories are being developed.

A great breakthough occured in 1997, when Maldacena discovered a beautiful relationship called the AdS/CFT correspondence [1], which today we see as a particular example of what is called the holographic duality or the gauge/gravity correspondence. The AdS/CFT correspondence is a duality between a QFT and gravity. Originally it related the physics of a 4-dimensional Conformal Field Theory (CFT), to the geometry of a 5-dimensional Anti de-Sitter (AdS) spacetime. It is also a duality between strongly and weakly coupled theories, i.e. it relates strongly coupled many-body quantum systems (such as, e.g. a quark-gluon plasma) with the dynamics of classical weakly coupled general relativity. In the past two decades, this correspondence has been extensively studied, leading to major developments in a broad range of areas, from black holes to condensed matter physics [2]. For instance, the AdS_3/CFT_2 correspondence led to the understanding of the microscopic origin of the BTZ black hole entropy [3]. In general, it is also a powerful tool to perform certain calculations of QFTs using relatively simple classical gravity techniques. One particular example is out-of-equilibrium physics of many body systems, which is generally not easy to deal with. Usually one would have to compute the density matrix of the system, and then be able to follow its time evolution. In practice, this is often not possible. However, using the duality one can compute this sort of things by solving Einstein equations, which are second order partial differential equations. Although sometimes they are non-trivial to solve, it is far more manageable than dealing with the density matrix.

This thesis can be divided into two parts. The first part, which consists on the first two articles, uses the holographic duality as a tool to calculate the evolution of quantum systems in non-equilibrium. In the first one we set up an holographic scenario to model what is called a quantum quench. In the second one, we study transport effects induced by the gravitational contribution to the chiral anomaly. The second part of the thesis, comprising the third article, focuses solely on general relativity. We analyze the asymptotic symmetries of non-expanding black holes and study the effect of the so-called horizon supertranslations on the their geometry.

After a short introduction to each of these topics, the articles are presented as they were published and finally, a short summary and conclusions of the results is given.

QUANTUM QUENCHES

In the past two decades, there has been exhaustive studies and lots of work on holographic theories and possible realizations of the holographic principle, in particular on the AdS/CFT correspondence. However, most of it has concerned static or stationary scenarios, and when it comes to the study of time dependent situations, much less is known.

There are several ways to bring a system out of equilibrium. Some examples include applying an external field, pumping in particles or energy, or changing a parameter of the system. We will focus on this last case, which receives the name of a quantum quench. A physical example of a quench would be to suddenly change the value of the magnetic field in a spin chain. This would create a situation in which all the spins will evolve and equilibrate to a new ground state in the modified hamiltonian. Over the last years, modeling quenches in a holographic setup has attracted considerable attention. A remarkable success has been achieved in reproducing important aspects of the universal dynamics of quenches [4–8]. However most models lack some of the defining characteristics of quenches. Notably, they simulate an injection of energy in the system without real change in the hamiltonian.

The aim of the first article of this thesis is to construct a simple holographic model of a quench modifying the infrared physics. To this end, we search for the gravitational dual to a family of QFTs parameterized by a relevant coupling. As a main input, the ground state for any value of the coupling is required to be Lorentz invariant. The simplest case involves Einstein gravity coupled to a real scalar field. In this picture, the process of equilibration of the QFT is dual to the collapse of a scalar shell of matter into a black hole, which represents the final equilibrium state. Therefore, the ground states of the family of QFTs will be represented by Anti de-Sitter black holes. We study these backgrounds and then construct the metrics modeling the quantum quench. Then, we numerically solve the equations of motion and analyze the phenomenology of possible outcomes after the quench.

ANOMALOUS TRANSPORT

Quantum anomalies, and in particular anomalous transport effects, have been an object of interest and research over the last years [9, 10]. When the classical lagrangian of a theory enjoys a certain symmetry which is not compatible at the quantum level we say that the theory is anomalous. Classically, due to Noether's theorem, there is a conserved current associated to this symmetry while at the quantum level this current conservation is violated. Moreover, if an anomalous symmetry is gauged, the theory ceases to be unitary, no longer making physical sense. The absence of gauge anomalies leads to constraints when constructing a quantum field theory, which is why, for instance, one needs 10 dimensions for (supersymmetric) string theory to work. A notable example of a quantum anomaly, and the one we will focus on, is the chiral anomaly. In this case, the symmetries of the classical theory are the independent phase rotations of left- and right-handed fermions. At the quantum level, you can have only the freedom to perform a linear combination of these two symmetries (see [10] for a good review). The chiral anomaly has a gauge component and a gravitational component (to which we will refer as gravitational anomaly). Explicitly, the non-conservation of the Noether current reads

$$\partial_\mu J^\mu = \epsilon^{\mu\nu\rho\sigma}(\alpha F_{\mu\nu} F_{\rho\sigma} + \lambda R^a_{b\mu\nu} R^b_{a\rho\sigma}) \tag{1}$$

where α and λ are some coefficients, $F_{\mu\nu}$ is the gauge field strength and $R^a_{b\mu\nu}$ is the Riemann tensor.

In the recent years, the chiral anomaly has allowed to understand and discover uncommon transport phenomena involving chiral fermions. The prime example is the so called *chiral magnetic effect* (CME) [11]. When chiral fermions are in the presence of a magnetic field, the chiral asymmetry generates a current proportional to the magnetic field

$$\vec{J} = \sigma \vec{B} \tag{2}$$

where σ is the anomalous transport coefficient and depends on the difference between right- and left-handed chemical potentials, $\sigma \sim \mu_5$. Moreover, at finite temperature T, the chiral fermions also build up an energy current [12, 13]

$$\vec{J}_\epsilon \sim T^2 \vec{B} \tag{3}$$

where, for simplicity, we have set the chemical potentials to zero. Remarkably, experimental transport signatures induced by (3) have been recently discovered in the Weyl semimetal NbP [14].

So far, anomaly induced transport has mostly been studied in near equilibrium scenarios, where local versions of temperature and chemical potentials can be defined. An application of anomaly induced transport is the quark gluon plasma created in heavy ion collisions [15]. There the magnetic field is extremely strong. It is also very short lived and might have already decayed before local thermal equilibrium is reached [16]. One therefore needs a better understanding of how anomalies induce transport far out of equilibrium. As we mentioned before, holography is an extremely efficient tool to study both anomalous transport phenomena and out of equilibrium dynamics of strongly coupled quantum systems. Furthermore, previous studies of anomalous transport in holographic quenches focused only on the purely gauge component of the chiral anomaly [17, 18]. In the second article of this thesis we use quantum quenches to bring the system far from equilibrium and study the anomalous transport induced by the gravitational anomaly.

ASYMPTOTIC SYMMETRIES OF BLACK HOLES

The second part of this thesis, which consists on the third article, focuses on the gravitational interaction. Needless to say, the way we understand gravity completely changed when Einstein presented his theory of General Relativity in 1905. In a nutshell, it says that matter and energy curve spacetime, while spacetime tells matter and energy how to move, and it can be mathematically expressed through the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{4}$$

where the left hand side represents the geometry of the spacetime and the right hand side, the matter and energy contained in it. This theory predicts one of the most fascinating objects in the universe, black holes. Black holes are regions of spacetime where matter is so dense and spacetime is so curved that not even light can escape its gravitational pull. Although at the center of a black hole there is a singularity of spacetime, its most interesting region is a surrounding outer null surface called the event horizon. Everything inside the horizon is forever doomed to remain there, and therefore a mystery for outside observers. Moreover, a black hole is only characterized by its asymptotic charges, its total mass, charge and angular momentum. This property goes under the name of the *no-hair theorem*.

Some years after the prediction of black holes, Bekenstein and Hawking found out that black holes also behave as thermodynamical objects, with an entropy and a temperature associated to them [19]. In a fascinating article, Hawking showed that black holes radiate energy away as if they were black bodies, that is, with an energy spectrum only characterized by the temperature of the radiating object [20]. This process is known as *Hawking radiation*, and can be understood in a simple way by the creation of quantum pairs of particles-antiparticles near the horizon, where one of the two falls into the black hole while the other one escapes to infinity. Thus, black holes are not eternal, they slowly evaporate and can in principle eventually disappear. Soon after these discoveries, Hawking himself realized that there existed a paradox with the thermodynamical nature of black holes [21]. Although black holes are described classically by the theory of General Relativity, it is believed that an underlying quantum description of them must exist, consistent with the not yet discovered quantum theory of gravity. One of the principles of Quantum Field Theory is that in every quantum process, information is not lost. However, this apparently contradicts the process of a black hole formation and evaporation. When a star is about to collapse, all the information about each of its particles positions and velocities is known. However, when the black hole is formed we cannot access this information due to the horizon barrier. Finally, in the last stage of the process the black hole radiates away all its energy, but this thermal radiation only knows about the temperature (or equivalently, the mass) of the black hole and therefore we seem to have lost all the information about the particles that originally formed the black hole. This is called the *black hole information paradox*, and has remained unsolved for more than 40 years now.

After this historical review, we now jump to two years ago, 2016, when Hawking, Perry and Strominger (HPS) proposed a possible solution to this problem, by studying the asymptotic symmetries of the black hole horizon [22, 23]. The idea comes from the fact that asymptotic symmetries of the boundary of spacetime have proven to be crucial in many ways. For instance, in a flat Minkowski spacetime, the asymptotic symmetry group (ASG) is the BMS[1] group [24–26], which is like the Poincaré group but with translations enhanced to an infinite-dimensional, abelian group called *supertranslations*. These supertranslations characterize the radiation that goes away (and comes from) future (past) null infinity, and they are in one-to-one correspondence with the infinite-dimensional space of radiative vacua of flat spacetimes, as was proven by Ashtekar in the early 80's [27–29]. In other words, a BMS supertranslation acts non-trivially on the geometry at null infinity, and transitively shifts from one radiative vacuum to another. Following the analogy with this case, it has been recently argued that one could enhance the ASG to the diffeomorphisms that leave invariant the near horizon geometry, now regarded as an inner boundary of spacetime [22, 23, 30–39]. For stationary black holes it was shown that, as in the case of null infinity, the ASG of the horizon is also enhanced with respect to the isometries of the background, by replacing translations with *horizon supertranslations*. HPS then conjectured that horizon supertranslations may act non-trivially on the black hole geometry, transforming the black hole to a physically inequivalent one[2]. Were this the case, some information could be encoded sort of holographically in these horizon supertranslations, at least partially resolving the black hole information problem.

Although these ideas are certainly appealing, the physical nature of the asymptotic symmetry group at the horizon is not clear, and the proposal remains controversial [40–42]. In our article, we tackle the problem in a different way from what has been done in the literature. We work in a framework which is coordinate independent, based on the works by Geroch [43] and Ashtekar [44, 45] on null infinity, and we are very careful to identify and characterize all the sources of gauge redundancy. On one hand, we have identified the group of horizon symmetries and checked that supertranslations belong to it and can be identified with a gauge redundancy of the description. However, with this result one cannot yet conclude that they act trivially on the black hole geometry, as they could be global symmetries (large gauge transformations) which can change the dynamical state of the system [46–48]. Therefore, on the other hand, we have identified the dynamical degrees of freedom of the horizon, and we have presented a free data set with the following properties

- It does not involve unfixed gauge redundancies
- It is both necessary and sufficient to reconstruct the black hole horizon geometry
- It is composed of objects which are invariant under horizon supertranslations

An immediate consequence, and our main conclusion, is that supertranslations do not affect the horizon geometry, and should be considered as a gauge redundancy of the description.

1 Named after Bondi, van der Burg, Metzner and Sachs

2 Note that this would not contradict the no-hair theorem since both black holes would still be diffeomorphic to each other.

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Collection of articles

1

A SIMPLE HOLOGRAPHIC SCENARIO FOR GAPPED QUENCHES

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A simple holographic scenario for gapped quenches

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Abstract: We construct gravitational backgrounds dual to a family of field theories parameterized by a relevant coupling. They combine a non-trivial scalar field profile with a naked singularity. The naked singularity is necessary to preserve Lorentz invariance along the boundary directions. The singularity is however excised by introducing an infrared cutoff in the geometry. The holographic dictionary associated to the infrared boundary is developed. We implement quenches between two different values of the coupling. This requires considering time dependent boundary conditions for the scalar field both at the AdS boundary and the infrared wall.

Keywords: Holography and condensed matter physics (AdS/CMT), AdS-CFT Correspondence

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Contents

1 Introduction

Modelling quantum quenches in a holographic setup has attracted considerable attention in the last years. A remarkable success has been achieved in reproducing important aspects of the universal dynamics of quenches [1]–[5]. However most models lack some of the defining characteristics of quenches. Notably, they simulate an injection of energy in the system without real change in the Hamiltonian.

In this note we want to present a simple holographic model of a quench modifying the infrared physics. With this aim, we search for the gravitational dual to a family of d-dimensional QFT's parameterized by a relevant coupling. As a main input, the ground state for any value of the coupling is required to be Lorentz invariant. We pursue the minimal scenario, involving Einstein gravity coupled to a real scalar field. The possibility of extra compactified dimensions is excluded.

We use the following ansatz for the ground state metrics

$$ds^2 = \frac{1}{z^2}\left(\frac{dz^2}{A(z)} - dt^2 + d\vec{x}^{\,2}_{d-1}\right) . \tag{1.1}$$

Setting $8\pi G = d-1$, the equations of motion are

$$\frac{z}{2}\,A' = d(A-1) + 2V(\phi)\,, \qquad A' = 2\,zA\phi'^2\,. \tag{1.2}$$

These equations have two integration constants, which can be related to the coefficients of the two independent scalar modes. Asking for regularity of the geometry links their values, allowing to interpret one of them as a QFT coupling and the other as the expectation value of the sourced operator. Regular solutions of (1.2), when they exist, describe RG flows between two fixed points associated to extrema of the scalar potential. The free integration constant represents then an irrelevant perturbation of the infrared fixed point, whose properties are determined by the scalar potential alone. All other solutions run into naked singularities. Considering naked singularities raises a number of serious problems.

JHEP02(2017)130

There is no condition that relates the two integration constants, challenging the usual holographic dictionary. Related to this, Lorentz invariant metrics are not minimal energy solutions when only one integration constant is fixed at the AdS boundary. Actually there are solutions of arbitrary negative energy.

These issues admit a simple, albeit crude solution, by introducing an infrared cutoff in the geometry. This creates a new boundary and renders it natural to interpret both integration constants from (1.2) as couplings. Fixing the two couplings solves the vacuum stability problem [6]. Moreover regions of high curvature, which bring outside the regime of validity of classical gravity, are excised.

AdS with an infrared hard wall is a well known rough holographic model for confining theories [7, 8]. The new ingredient in this paper is to consider the hard wall as a regularizing element, while the infrared physics will be linked to the strength of the naked singularity.

This paper is organized as follows. In section 2 we solve the equations of motion for the static backgrounds with a vanishing potential and study the first harmonic modes. In section 3 we model a quantum quench between two values of the coupling and then, in section 4, we analyze the numerical results. Finally, the holographic dictionary associated to the infrared boundary is developed in section 5, where we also briefly summarize our work.

2 Static backgrounds

We will explore the proposed scenario with a vanishing scalar potential. Equations (1.2) can then be solved analytically, with the result

$$A(z) = 1 + \alpha^2 z^{2d} \,, \tag{2.1}$$

$$\phi(z) = \beta + d^{-1/2} \operatorname{arcsinh}(\alpha z^d) \,, \tag{2.2}$$

where α and β are arbitrary constants. β represents a global shift in the value of the scalar, which is of no physical consequence when $V(\phi)=0$. α induces a non-trivial scalar profile

$$\alpha = z_0^{-d} \sinh(\sqrt{d}\, \Delta\phi) \,. \tag{2.3}$$

with z_0 denoting the radial position of the wall, and $\Delta\phi = \phi_0 - \phi_\infty$ the variation of the scalar field between the wall and the AdS boundary. If extended beyond the infrared cutoff, all backgrounds with $\Delta\phi \neq 0$ have a naked singularity. Their curvature is

$$R(z) = -d(d+1) + d(d-1)\alpha^2 z^{2d} \,. \tag{2.4}$$

Holography interprets the harmonic modes of bulk fields as excitations in the dual QFT. In figure 1 we have plotted the frequency of the lower scalar modes along the family (2.1)–(2.2) for $d=2$. When the radial variation of scalar profile is small, the spectrum is determined by z_0 . The spectrum becomes instead ruled by $\Delta\phi$ for larger values of this parameter. Figure 1a shows that the mass gap, holographically given by ω_0, grows with $\Delta\phi$ and implies that this is a relevant coupling. Interestingly the ratio of higher normal frequencies to the fundamental one shows an approximate linear growth, see figure 1b. Hence the infrared physics associated to the family (2.1)–(2.2) does not differ by a mere rescaling. The lowest excitation becomes increasingly separated from the rest the larger is $\Delta\phi$.

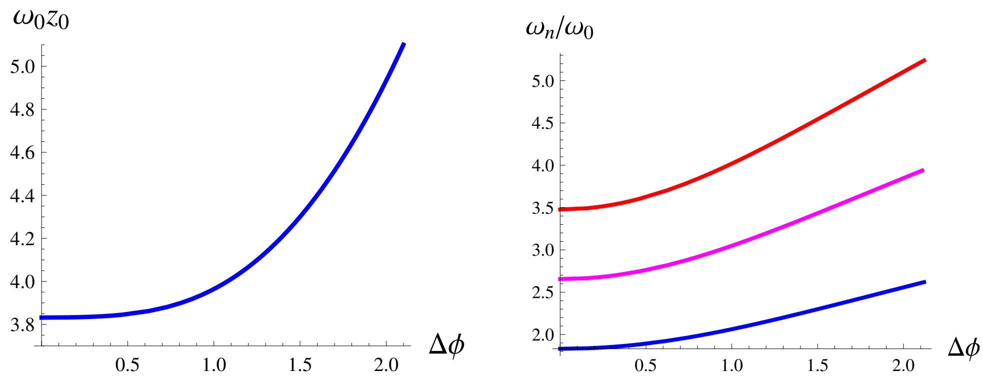

Figure 1. Left: frequency of the fundamental scalar mode. Right: ratio of the next three normal frequencies to the fundamental one.

3 Modelling a quantum quench

We want to model a global quench between QFT's whose ground states are in the family (2.1)–(2.2). A convenient ansatz for the associated metric is

$$ds^2 = \frac{1}{z^2}\left(-A(t,z)e^{-2\delta(t,z)}dt^2 + \frac{dz^2}{A(t,z)} + d\vec{x}^2_{d-1}\right) . \tag{3.1}$$

For vanishing scalar potential, the equations of motion are

$$\dot{\Phi} = \left(Ae^{-\delta}\Pi\right)' , \qquad \dot{\Pi} = z^{d-1}\left(z^{1-d}Ae^{-\delta}\Phi\right)' , \tag{3.2}$$

$$\delta' = z\,(\Phi^2 + \Pi^2)\,, \qquad A' = z\,A\,(\Phi^2 + \Pi^2) + \frac{d}{z}(A-1)\,, \tag{3.3}$$

with $\Phi = \phi'$ and $\Pi = A^{-1}e^{\delta}\dot{\phi}$ encoding the radial and time scalar derivatives. Solving the equations of motion requires giving a set of initial data together with boundary data at asymptotic AdS and the infrared wall. As boundary data, we will allow for time dependent profiles $\phi_\infty(t) = \phi(t,0)$ and $\phi_0(t) = \phi(t,z_0)$. Dynamical processes triggered by $\phi_\infty(t)$ in the hard wall setup were studied in [9, 10]. The possibility of imposing a time dependent scalar profile at the wall has been considered in [11].

The family (2.1)–(2.2) does not exhaust the set of static solutions to our gravity system. In general they break Lorentz invariance and are described by the ansatz (3.1). Their energy density can be read from the asymptotic expansion $A = 1 - 2Mz^d + \ldots$. It is then clear that all solutions (2.1)–(2.2) have zero mass. For static solutions

$$M = \frac{1}{2}z_0^{-d}(1-A_0) + \frac{1}{2}\int_0^{z_0} z^{1-d}A\Phi^2 dz\,, \tag{3.4}$$

with $A_0 = A(z_0)$. The first term represents the contribution to the total energy from the geometry hidden by the infrared cutoff. When this part encloses a naked singularity it can be arbitrarily negative. On the contrary, the second term is always positive for solutions without horizons. Therefore naked singularities are a crucial ingredient for obtaining Lorentz invariant backgrounds with non-trivial scalar profiles.

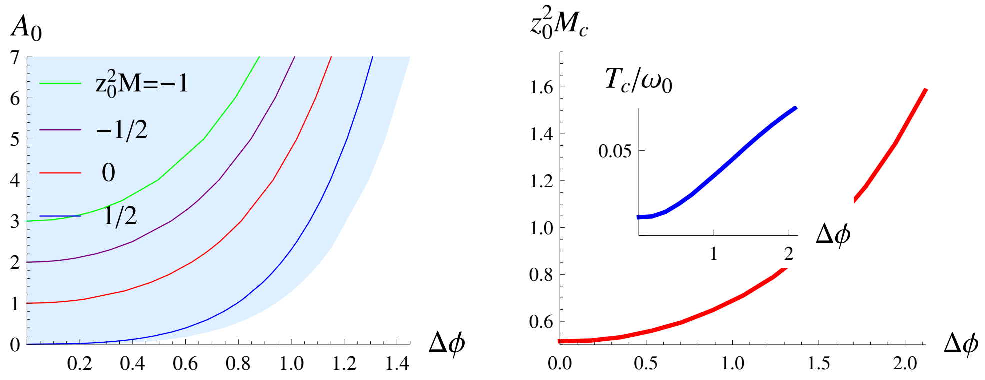

Figure 2. Left: shadowed in blue, couplings admitting static solutions without horizons for $d=2$. Equal mass curves are highlighted. Right: threshold mass for collapse of pulses (4.1). Inset: ratio of the temperature of the black hole at threshold to the mass gap.

Up to a trivial global shift in the scalar, static solutions are parameterized by $\Delta\phi$ and A_0. The backgrounds (2.1)–(2.2) define the codimension one subset

$$A_0 = \cosh^2(\sqrt{d}\,\Delta\phi)\,, \tag{3.5}$$

see red line in figure 2. Since A_0 is a boundary data, it is natural to also interpret it as a QFT coupling. The unique static solution without horizons for $\Delta\phi$ and A_0 in the shaded region of figure 2a, represents the ground state of the associated QFT [11].

We consider that the QFT before the quench is in the ground state for chosen couplings in the subset (3.5). Acting on the boundary values such that ϕ_∞ changes while ϕ_0 remains constant, clearly brings outside (3.5). The same actually happens in the opposite case. When ϕ_∞ is kept constant, the equations of motion ensure the conservation of total mass. A time dependent ϕ_0 generates a scalar pulse that enters the geometry at the infrared boundary. Unless the time variation is adiabatic, this pulse induces an excited state in the final QFT. This injection of energy is countered by an adjustment of the value of A_0 such that the total energy is conserved, bringing again outside (3.5). Namely, if the initial theory belongs to the Lorentz invariant subset, the final one will have negative ground state energy. Therefore, the only way to model a quench between theories in (3.5) is by a combination of the wall profile $\phi_0(t)$ with a net energy injection into the system. This can only happen at the AdS boundary, induced by a non-trivial $\phi_\infty(t)$. The final configuration will be an excited state above a new, but still Lorentz invariant, ground state.

The initial state will be taken to have vanishing ϕ_∞, $\phi_0 = \bar{\phi}$ and A_0 satisfying (3.5). We shall choose the wall profile

$$\phi_0(t) = \bar{\phi} + \frac{1}{2}\eta\left(1 + \tanh\frac{t}{a}\right). \tag{3.6}$$

This models a quench with a finite time span controlled by the parameter a. After the quench $\phi_0 = \bar{\phi}+\eta$, while A_0 will be fixed by the conservation of energy at the wall. Any $\phi_\infty(t)$ which fulfills (3.5) at late times, ensures that the final QFT will have Lorentz invariant couplings. We do not want however that the quench follows an arbitrary path in the coupling space of figure 2a. We aim to only act on the combined coupling that moves along the $M = 0$ line. This involves a tuned variation of the scalar field at the wall and the AdS boundary, which the absence of time-like Killing vector renders it unclear how to implement. In the following we will assume that the diagonal time coordinate in (3.1) provides a reasonable way to project the value of ϕ_0 onto the dual QFT. Hence we require (3.5) to hold at each constant time slice.

We use a Runge-Kutta algorithm of fourth order to solve the evolution equations (3.2), as done in [11]. The time coordinate t is gauge fixed to be the proper time at the AdS boundary, i.e. $\delta_\infty = 0$. The boundary conditions are numerically implemented in the code by imposing

$$\Pi_0 = A_0^{-1} e^{\delta_0} \dot{\phi}_0 \,, \qquad \Pi_\infty = \dot{\phi}_0 \left(1 - z_0 \Phi_0 \sqrt{\frac{A_0}{d(A_0 - 1)}} \right) , \tag{3.7}$$

with ϕ_0 in (3.6) and Π_∞ derived from (3.5). The wall values of A, δ and Φ are fixed by the equations of motion in terms of ϕ_0 and the initial conditions.

4 Numerical results

The central characteristic of the dynamics generated by (3.5)–(3.6), is whether or not it will generate a horizon. In the affirmative case, the end point of the evolution is a Schwarzschild black hole trapping the total mass. This represents a unitary process in the dual QFT leading to thermalization [1, 12]. Those that do not form a horizon result in a scalar pulse that bounces forever between AdS boundary and wall [9, 10]. Bouncing geometries provide the holographic counterpart to periodic reconstructions of quantum correlations in the dual field theory [13, 14], known as quantum revivals [15].

The only topological obstruction to the formation of a horizon in our setup is the presence of the wall, enforcing that the total mass of the configuration satisfies $M z_0^2 > 1/2$. A first question then is whether the typical scale triggering fast thermalization is set by z_0 or depends on the $\Delta\phi$. Before studying quenches, we analyze the infall of a scalar shell modelling an energy injection without variation of the Hamiltonian. We restrict in the following to $d = 2$ for numerics. Since the shape of the pulse influences the evolution, we consider a typical shell, radially localized and of gaussian form

$$\Pi(t=0) \propto z^2 e^{-\frac{1}{\sigma^2} \tan^2\left(\frac{\pi z}{2 z_0}\right)} , \tag{4.1}$$

with $\sigma = 0.1$ and $\Phi(t=0)$ in the family (2.1)–(2.2). The threshold mass for gravitational collapse without bounces is plotted in figure 2b. It strongly grows with $\Delta\phi$, confirming the secondary role of the infrared wall. Using the hard wall as an auxiliary element, we are actually obtaining a basic model of a soft wall.

We explore now the evolutions after a quench modelled by (3.5)–(3.6) in $d = 2$. The quench will be applied to the Lorentz invariant background $\Delta\phi = \bar{\phi} = 0.7$. At this value the infrared physics starts to be dominated by the hidden singularity instead of the wall

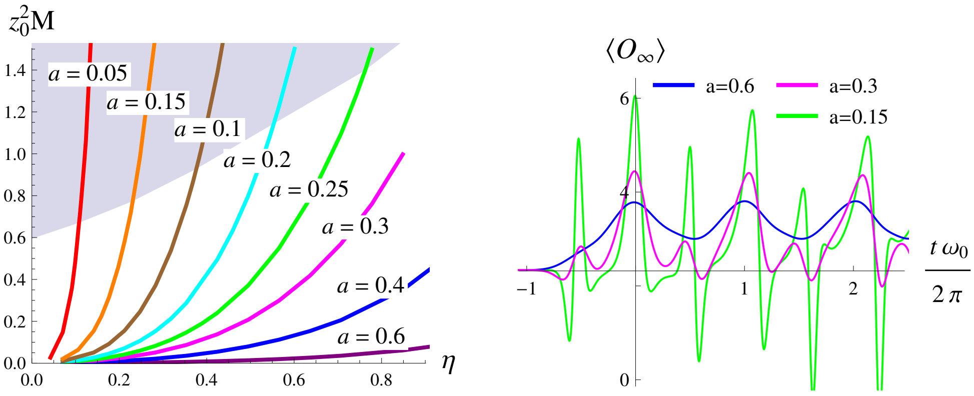

Figure 3. Left: energy density generated by (3.5)–(3.6) for $\bar{\phi} = 0.7$ and $d = 2$. Right: $\langle \mathcal{O}_\infty \rangle$ for three quenches with different time spans.

position, see figure 1. We focus on $\eta > 0$, and hence the quench will increase the mass gap. Figure 3a shows the final energy density as a function η for several values of the time span a. Its growth with η is more pronounced the smaller is a. We have shaded in blue the parameters that lead to black hole formation. Processes where a horizon is generated after some bouncing cycles occupy just a small window on the boundary of the blue region. Otherwise we obtain geometries that keep bouncing as far as our simulation could go. Only sufficiently fast quenches, those with $a < 0.25$ in the example of figure 3a, can generate enough energy density to trigger thermalization.

Bouncing geometries can be roughly divided in two types: standing and traveling waves. Standing waves project mainly on the fundamental harmonic of the static background associated to the final couplings. It is convenient to restore the natural mass units, $M \to \frac{d-1}{8\pi G} M$, with G extremely small. According to the holographic dictionary, $1/G$ is proportional to the number of elementary degrees of freedom in the dual QFT. Hence M translates into an energy density per species in field theory terms. Although the mass of standing waves is much smaller than that required for collapse, it can be parametrically larger than G. Indeed quenches in figure 3a generate standing waves when $a \geq 0.6$, having masses up to $M z_0^2 \approx 0.1$.

Standing waves oscillate with the frequency of the mass gap, ω_0. It is then natural to holographically identify them with coherent states of $\vec{k} = 0$ modes of the lowest QFT excitation. Revivals with the same interpretation appear for example in the massive Schwinger model after a quench [16, 17]. The important difference in our case is their energy density. It can be much larger than the mass gap, proper in holographic models of a confining phase, ranging up to $O(1/G)$, close to the typical values in the plasma phase. In spite of that, the physics driving thermalization does not refer to ω_0. This is illustrated in the inset of figure 2b. The temperature of the black hole at the collapse threshold for the gaussian pulses (4.1), is well below the mass gap.

Traveling pulses exhibit radial localization and displacement. They represent in general partial revivals. They have larger masses, and the associated QFT states are thus expected to contain higher energy excitations and non-zero momentum modes. The former should be connected with the projection of narrow pulses on higher harmonic modes. The radial infall of a narrow shell has been related to the evolution of the separation between entangled excitations after a quench [1, 3, 5], the so-called horizon effect [18]. In this sense, radial displacement indicates the presence of non-zero momentum modes in the dual field theory state. Since the quench we are modelling is global, finite momentum modes can only be created in pairs. Figure 1b shows that $2\omega_0 \approx \omega_1$ for a large range of couplings, explaining why radial localization and displacement appear at similar energies.

Contrary to standing waves, traveling configurations generated by (3.5)–(3.6) are composed of two distinct sub-pulses, one entering from the AdS boundary and the other from the wall. This is clearly appreciated in the one-point functions. Figure 3b shows the vev of the operator sourced by ϕ_∞ for three examples from figure 3a. We use a rescaled time such that the fundamental frequency for the final couplings is 2π. The oscillations of $\langle \mathcal{O}_\infty \rangle$ are plotted in blue for a slow quench, with $a = 0.6$, resulting in a standing wave. A traveling configuration with two sub-pulses producing signals of similar magnitude is obtained for $a = 0.15$ and shown in green. The effect of both subpulses superposes, giving rise to oscillations with roughly twice the fundamental frequency. In magenta we have an intermedium configuration, with a small boundary component. It is worth mentioning a slight increase in the oscillation period of each subpulse between the $a = 0.6$ and $a = 0.15$ pulses. This is due to their different final energies: $M = 0.002$ and $M = 0.02$ respectively. The increase of the period with the energy is generic in holographic quenches, finding some analogues in the condensed matter literature [13, 14].

The distinction between fast and slow quenches should refer to the characteristic scale of the infrared physics. Slow quenches can be unambiguously defined as those producing standing or quasi-standing waves. We consider now quenches with fixed amplitude η and time span a, but different initial coupling $\bar{\phi}$. Figure 1a shows that the mass gap grows with the coupling. Therefore the quench should result in a collapsing shell, a bouncing pulse or a standing wave as we choose larger values of $\bar{\phi}$. Alternatively, the energy density in units of the final mass gap must be a monotonically decreasing function of $\bar{\phi}$. This quantity is plotted in figure 4a for $\eta = 0.2$ and several small values of a, confirming the expected behavior.

5 One point functions at the wall

We have assumed that the boundary values ϕ_0 and A_0 relate to couplings with a well defined, local projection on the field theory time coordinate. The same as ϕ_∞, they should source local operators. We aim to determine their expectation values.

Symmetry under global shifts of the scalar field implies that only the difference $\Delta\phi = \phi_0 - \phi_\infty$ is physically relevant. Hence the ground state expectation values of the operators $\mathcal{O}_0$ and $\mathcal{O}_\infty$ can not be independent. While the latter is dictated by the asymptotic expansions at the AdS boundary, the former has to depend on quantities evaluated at the wall. The scalar equation for static solutions reduce to $(z^{1-d} A e^{-\delta} \Phi)' = 0$, implying

$$\lim_{z\to 0} (z^{1-d}\Phi) = z_0^{1-d} A_0 e^{-\delta_0} \Phi_0 \,. \tag{5.1}$$

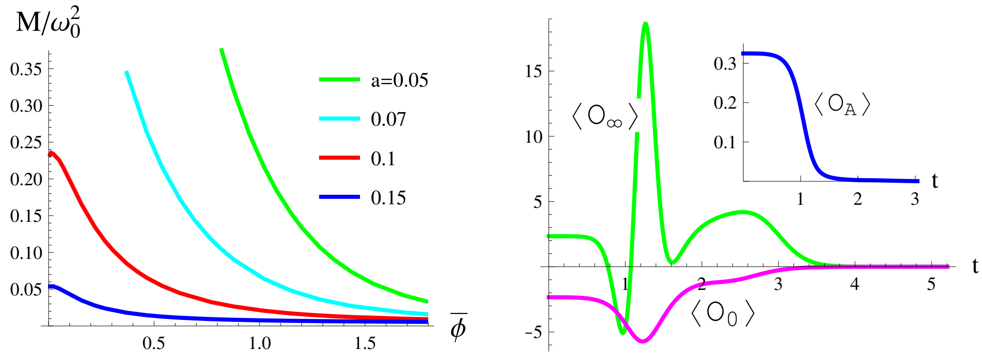

Figure 4. Left: energy density normalized by the square of the final mass gap in quenches with different initial coupling and $\eta = 0.2$. Right: evolution of one-point functions after a quench with $a = 0.3$ and $\eta = 1$, leading to thermalization.

where we have used $A_\infty = 1$ and the gauge choice $\delta_\infty = 0$. The l.h.s. is precisely $\langle \mathcal{O}_\infty \rangle$ [19]. Defining $\langle \mathcal{O}_0 \rangle$ as minus the r.h.s., we obtain a relation of the desired form

$$\langle \mathcal{O}_\infty \rangle + \langle \mathcal{O}_0 \rangle = 0 \,. \tag{5.2}$$

The sign has been chosen such that the operator sourced by $\phi_\infty + \phi_0$ has a vanishing vev in the ground state. Notice that (5.1) would not hold with $V(\phi) \neq 0$, when neither a global shift on the scalar is a symmetry of the system.

The metric function A satisfies the evolution equation

$$\dot{A} = 2zA\Phi\dot{\phi} \,. \tag{5.3}$$

The z^d coefficient in the asymptotic expansion of A determines the dual QFT energy density [19]. The previous equation implies $\dot{M} + \dot{\phi}_\infty \langle \mathcal{O}_\infty \rangle = 0$. However the field theory Ward identities dictate a sum over all couplings, $\dot{M} + \sum \dot{\lambda}_i \langle \mathcal{O}_i \rangle = 0$ [19]. It is then necessary that the contributions from ϕ_0 and A_0 exactly cancel, which is the requirement of energy conservation at the wall. Using the above proposed value for $\langle \mathcal{O}_0 \rangle$, (5.3) at the wall can be rewritten as

$$\dot{\phi}_0 \langle \mathcal{O}_0 \rangle + \frac{1}{2} \dot{A}_0 \, z_0^{-d} e^{-\delta_0} = 0 \,. \tag{5.4}$$

Therefore the expectation value of the operator $\mathcal{O}_A$ sourced by A_0, is given by the expression multiplying its time derivative in the previous equation.

A check on the consistency of these assignments is how they behave when a horizon forms. Thermalization after a global quench in an infinite system only happens at the local level. Namely, for any late but finite time there are sufficiently large regions where non-local observables have not yet achieved thermal values. Such observables, as for example the entanglement entropy, require information from behind the apparent horizon for their holographic determination [1, 4]. One point functions are local observables, which thus should only imply the geometry outside it. We have used constant t slices to translate wall boundary values into QFT couplings. Constant t slices only approach the apparent horizon asymptotically at late times, in the region where it has practically achieved its final value z_{BH}. They depart again from it at $z > z_{BH}$, and finally reach the wall. This implies that indeed, $\langle \mathcal{O}_0 \rangle$ and $\langle \mathcal{O}_A \rangle$ do not require information from behind the apparent horizon at any instance of their evolution.

The only non vanishing one-point function associated to a Schwarzschild geometry is that of the stress tensor. Thus other expectation values should tend to zero in the process of gravitational collapse. When a horizon emerges, the part of the geometry with $z > z_{BH}$ gets frozen for observers using the proper time at the AdS boundary. This is implemented by the exponential vanishing of $e^{-\delta}$ in that region. According to the previous assignments both $\langle \mathcal{O}_0 \rangle$ and $\langle \mathcal{O}_A \rangle$ are proportional to $e^{-\delta_0}$, which insures that indeed they tend to zero as a horizon forms. Clearly so does $\langle \mathcal{O}_\infty \rangle$. The evolution of the three observables after a quench generating a horizon, or equivalently leading to thermalization, is shown in figure 4b.

We have proposed a simple holographic scenario, easily accessible to numerics, modelling quenches where a relevant coupling changes. A number of checks have been successfully performed. We hope that this can help in placing holography among the standard tools for studying out of equilibrium physics.

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2

QUENCHING THE CHIRAL MAGNETIC EFFECT VIA THE GRAVITATIONAL ANOMALY AND HOLOGRAPHY

This chapter contains the article

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Quenching the Chiral Magnetic Effect via the Gravitational Anomaly and Holography

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In the presence of a gravitational contribution to the chiral anomaly, the chiral magnetic effect induces an energy current proportional to the square of the temperature in equilibrium. In holography the thermal state corresponds to a black hole. We numerically study holographic quenches in which a planar shell of scalar matter falls into a black hole and raises its temperature. During the process the momentum density (energy current) is conserved. The energy current has two components, a nondissipative one induced by the anomaly and a dissipative flow component. The dissipative component can be measured via the drag it asserts on an additional auxiliary color charge. Our results indicate strong suppression very far from equilibrium.

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Anomaly induced transport phenomena have been in the focus of much research in recent years [1,2]. The prime example is the so called chiral magnetic effect (CME) [3]. The CME has a component due to the gravitational contribution to the chiral anomaly. Chiral fermions in the background of gauge and gravitational fields have the anomaly

$$\partial_\mu J^\mu = \varepsilon^{\mu\nu\rho\sigma}(\alpha F_{\mu\nu}F_{\rho\sigma} + \lambda R^a_{b\mu\nu}R^b_{a\rho\sigma}). \tag{1}$$

In the presence of a magnetic field $\vec{B}$ and at temperature T chiral fermions build up an energy current [4,5]

$$\vec{J}_\varepsilon = 32\pi^2 T^2 \lambda \vec{B}. \tag{2}$$

In holography the gravitational anomaly is implemented via a mixed Chern-Simons term. Space-time is curved in the additional holographic direction and this is what generates Eq. (2) from the mixed Chern-Simons term [6]. Perturbative nonrenormalization has been shown in Refs. [7,8]. The relation of Eq. (2) with the gravitational contribution to the chiral anomaly has also been derived in a model independent manner combining hydrodynamic and geometric arguments in Refs. [9,10]. An additional constraint on effective actions consistent with Eq. (2) stems from considerations based on global gravitational anomalies [11,12]. Transport signatures induced by Eq. (2) have recently been reported in the Weyl semimetal NbP [13].

So far anomaly induced transport has mostly been studied in a near equilibrium setup in which local versions of temperature and chemical potentials can be defined. An application of anomaly induced transport is the quark gluon plasma created in heavy ion collisions [14]. There the magnetic field is extremely strong. It is also very short lived and might have already decayed before local thermal equilibrium is reached [15]. One therefore needs a better understanding of how anomalies induce transport far out of equilibrium. Holography is an extremely efficient tool to study both anomalous transport phenomena and out of equilibrium dynamics of strongly coupled quantum systems. Previous studies of anomalous transport in holographic quenches focused on the pure $U(1)^3$ anomaly [16,17]. This motivates us to study anomalous transport induced by the gravitational anomaly in a holographic quantum quench.

To develop intuition we first consider anomalous hydrodynamics [18–20] in a spatially homogeneous magnetic field. The anomalous transport effect we want to monitor is the generation of an energy current in the magnetic field (2). We start with an initial state at temperature T_0 and heat the system up to a final temperature T. In a relativistic theory the energy current $J^\varepsilon_i = T_{0i}$ is the same as the momentum density $\mathcal{P}_i = T_{i0}$. Momentum is a conserved quantity and cannot increase if it is not injected into the system. On the other hand the anomaly induced energy current does increase and this increase must be balanced by a collective flow not included in Eq. (2). The increase in the nondissipative anomalous energy current (2) will be exactly counterbalanced by a dissipative contribution at any given moment. If we introduce a uniform but very light density of impurities carrying some additional charge drag will generate a convective current proportional to the density of impurities [21]. This current measures how much energy current is generated via Eq. (2). We thus need to consider a system with two $U(1)$ charges. The first one has the mixed gravitational anomaly (1) and carries the magnetic flux. The second one is a anomaly free auxiliary $U(1)$ charge that serves to monitor the buildup of the current (2) as the temperature changes.

We now consider the hydrodynamics of the system. Since all spatial gradients vanish the constitutive relations are

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu + p\eta_{\mu\nu} + \hat{\sigma}_B(u_\mu B_\nu + u_\nu B_\mu), \quad (3)$$

$$J_\mu = \rho u_\mu + \sigma_B B_\mu, \quad (4)$$

$$J^X_\mu = \rho_X u_\mu + \sigma_{B,X} B_\mu. \quad (5)$$

The specific forms of the anomalous transport coefficients σ_B, $\hat{\sigma}_B$, and $\sigma_{B,X}$ depend on the hydrodynamic frame choice [23]. In the Landau frame

$$\hat{\sigma}_B = 0, \quad (6)$$

$$\sigma_B = 24\alpha\mu - \frac{\rho}{\varepsilon + p}(12\alpha\mu^2 + 32\lambda\pi^2 T^2), \quad (7)$$

$$\sigma_{B,X} = -\frac{\rho_X}{\varepsilon + p}(12\alpha\mu^2 + 32\lambda\pi^2 T^2). \quad (8)$$

Note that despite J^X_μ having no anomaly it does have a nontrivial chiral magnetic transport coefficient in the Landau frame. This makes the effect due to the dragging manifest. We assume that the system is neutral with respect to the anomalous charge, i.e., $\mu = \rho = 0$. As the initial condition we take $\vec{J} = \vec{J}^X = 0$ and the energy current to be given by Eq. (2). Thus, the energy current at any given moment is

$$32\lambda\pi^2 T_0^2 \vec{B} = (\varepsilon + p)\vec{v}, \quad (9)$$

where we work in the linear response regime; i.e., we assume λB to be a small perturbation compared to the energy density and pressure such that $u_\mu \approx (1, \vec{v})$. Solving for the velocity $\vec{v}$ and using the constitutive relation for the current $\vec{J}_X$ we find that it is given by

$$\vec{J}_X = 32\frac{\rho_X}{\varepsilon + p}(T_0^2 - T^2)\pi^2\lambda\vec{B}. \quad (10)$$

This equation determines the current buildup due to drag if the system undergoes a slow near equilibrium time evolution such that an instantaneous temperature T can be defined. It is independent of the choice of the Landau frame. For a generic conformal field theory $p = KT^4$ and $\varepsilon = 3p$, and we obtain

$$j_X = \frac{T^2/T_0^2 - 1}{T^4/T_0^4} \quad (11)$$

with $j_X = [(KT_0^2|\vec{J}_X|)/(8\pi^2|\rho_X\lambda\vec{B}|)]$. If a system undergoes near equilibrium evolution j_X must lie for any given moment on that curve. Deviation from Eq. (11) will be a benchmark for far from equilibrium behavior.

We will now consider a holographic model that allows us to implement the physics described in a nonequilibrium setting. The action of our model is

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\bigg(\mathcal{R} - 2\Lambda - \frac{1}{4}F^2 - \frac{1}{4q^2}F_X^2 - \frac{1}{2}(\partial\phi)^2 + \lambda\varepsilon^{MNOPQ}A_{Ms}R^A_{BNO}R^B_{APQ}\bigg). \quad (12)$$

In addition to gravity and a scalar field ϕ, it involves two gauge fields A_M and X_M with field strengths $F = dA$ and $F_X = dX$. Only gauge transformations of A are anomalous. The Chern-Simons term is the holographic implementation of the mixed chiral-gravitational anomaly [6,24]. We do not include possible pure gauge Chern-Simons terms since we want to isolate the effects of the gravitational anomaly. We set from now on $2\kappa^2 = 1$.

The dual field theory will be brought out of equilibrium by abruptly varying a coupling. These type of processes are known as quantum quenches [25]. Holographically, the leading mode of bulk fields at the holographic boundary is interpreted as a coupling for the dual field theory [26–28]. We will perform a quench on the coupling associated with the scalar ϕ. For simplicity, we have chosen ϕ to be massless and neutral under the gauge fields A_M and X_M. We consider the following simple boundary profile for its leading mode

$$\phi_0(t, \vec{x}) = \frac{1}{2}\eta\left(1 + \tanh\frac{t}{\tau}\right), \quad (13)$$

Since Eq. (12) is invariant under global shifts of ϕ, the Hamiltonians before and after the quench will be equivalent. The energy density induced by the quench is a function of both its amplitude η and time span τ.

In order to be able to match to the hydrodynamic results we will compute the linear response to the magnetic field, which is a valid approximation when $T_0 \gg \sqrt{B}$. It is thus enough to solve the equations of motion to first order both in the Chern-Simons coupling and the magnetic field. Besides, we will work in the decoupling limit of large charge q for the anomaly free gauge field. Equivalently, we can work at $q = 1$ and treat X^μ perturbatively to first order. In that case its dynamics does not backreact onto the other sectors of the theory.

We parametrize the zero-order solution of the metric as

$$ds^2 = \frac{1}{z^2}\left(-f(t,z)e^{-2\delta(t,z)}dt^2 + \frac{dz^2}{f(t,z)} + d\vec{x}^2\right). \quad (14)$$

The equations of motion for the background of gravity and the scalar sector are

$$\delta = \frac{1}{3}zT^\phi_{zz}, \qquad f' = \frac{4}{z}(f - 1) + f\delta, \quad (15)$$

$$\partial_t\left(\frac{e^\delta\partial_t\phi}{f}\right) = z^3\partial_z\left(\frac{fe^{-\delta}\partial_z\phi}{z^3}\right). \quad (16)$$

These equations are solved numerically using a fourth order Runge-Kutta algorithm. Time reparametrizations are used to fix $\delta(t, 0) = 0$, such that the Minkowski metric is reproduced at the holographic boundary and the bulk energy momentum tensor of the scalar field is $T^{\phi}_{MN} = \partial_M\phi\partial_N\phi - \frac{1}{2}(\partial\phi)^2 g_{MN}$.

Field theories at thermal equilibrium are dual to black hole backgrounds. Our initial geometry will be a Schwarzschild black hole, for which $f = 1 - \pi^4 T_0^4 z^4$ and $\delta = 0$. Around $t = 0$ the quench takes place, creating an extra energy density on the boundary theory and, equivalently, a matter shell that enters from the holographic boundary into the bulk. The shell subsequently undergoes gravitational collapse and is absorbed by the original black hole. The resulting black hole represents the final equilibrium state with $T > T_0$. Similar holographic setups have been extensively studied in the last years [29–32].

The gauge fields A_M and X_M satisfy the same equation at zero order in λ. However, the different roles they should fulfill select different leading solutions. We want A_M to induce a constant background magnetic field on the boundary theory and no charge density. Choosing the magnetic field to point in the x_3 direction, this is achieved by the simple solution $F_{12} = B$ with B constant throughout the bulk. To the contrary, we wish X_M to induce a nonvanishing charge density and no boundary field strength, which leads to

$$F_{X,0z} = \rho_X z e^{-\delta(t,z)}. \tag{17}$$

The expectation value of its associated current is given by

$$J^{\mu}_X = \lim_{z\to 0} \sqrt{-g}\, F^{\mu z}_X. \tag{18}$$

The integration constant ρ_X is the desired charge density.

At linear order in λ, the magnetic field induces a nondiagonal component in the metric

$$g_{03} = \frac{4\lambda B}{z^2} \int_0^z z^3 e^{-\delta} \left(c + \frac{12}{z^2}(f-1) + T^{\phi}_{33} \right), \tag{19}$$

where c is an integration constant. In the initial and final state the bulk scalar field stress tensor vanishes. We fix the integration constant by demanding that the initial state reproduces Eq. (2), which implies $c = 8\pi^2 T_0^2$. In our conventions the energy current can be read off the asymptotic metric expansion as $g_{03} = \frac{1}{4} T_{03} z^2 + \cdots$ [33].

Neither the scalar field and nor the anomalous gauge field receives a correction at order λ. However the off-diagonal component of the metric backreacts on X_M and generates an entry parallel to the magnetic field. It is governed by the equation

$$\partial_t \left(\frac{e^{\delta}\partial_t X_3}{zf} \right) - \partial_z \left(\frac{e^{-\delta} f \partial_z X_3}{z} \right) = \rho_X \partial_z (z^2 g_{03}). \tag{20}$$

Although X_3 is sourced by the Chern-Simons term even in the initial state, it leads to a vanishing J^3_X. It is the subsequent evolution that generates a boundary current parallel to the magnetic field. Integrating Eq. (20) in the final state with temperature $T > T_0$ we recover Eq. (11) upon normalizing the holographic current (18) accordingly. We also note that due to the fact that J^3_X is treated in the decoupling limit it reacts immediately to the drag.

We focus first on fast quenches. These are processes whose time span is small in units of the inverse final temperature, $2\tau T < 1$. Figure 1(a) shows the result of two such processes with the same initial temperature T_0 and time span τ but different final temperatures. A comparison with the benchmark curve (11), plotted in the lower inset of Fig. 1(a), rules out a possible near equilibrium description. In the hydrodynamic process the current j_X attains the maximum when the temperature reaches $T_m/T_0 = \sqrt{2}$, at which it takes the value 1/4. For higher temperatures the equilibrium current decreases, reflecting the large inertia of a hot medium. One of the processes in Fig. 1(a) has been tuned to reach T_m, while the other generates a higher temperature $T/T_0 = 3$. In both cases j_X exhibits a maximum before stabilizing. This contradicts Eq. (11), which would predict a monotonic growth for the process whose final temperature is T_m, and shows that the evolution is far from equilibrium. Moreover, the current at the maximum is larger than 1/4 in the first quench (blue) and smaller in the second (red), which again disagrees with Eq. (11). When the currents of both processes are normalized to 1 at the final equilibrium state, their profiles in the rescaled time tT are very similar. The current overshoots by around 4% its final value before attaining it, as can be seen in the upper inset of Fig. 1(a). The time span of a fast quench also has a very small impact on the evolution of the current. This is illustrated in the inset of Fig. 1(b), which shows three processes with the same final temperature $T/T_0 = 2.5$ and a different time span.

It is interesting to compare the evolution of j_X with that of another important observable, the energy density. The energy density builds up during the quench and attains its final value as soon as the quench ends. In terms of the

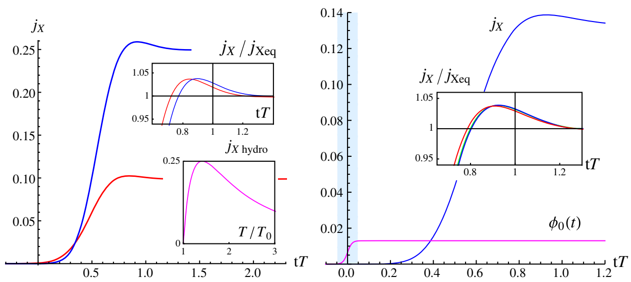

FIG. 1. Left: evolution of j_X for quenches with $\tau T_0 = 0.07$ and $T/T_0 = \sqrt{2}$ (blue) and 3 (red). The final current has been normalized to 1 in the upper inset. The lower inset shows the equilibrium curve (11). Right: very fast quench with $\tau T_0 = 0.007$ and $T/T_0 = 2.5$. The scalar profile (13) is included, shading the time span of the quench. The inset compares this process to two more quenches with $\tau T_0 = 0.0175$, 0.035, and the same T/T_0.

holographic model the energy pumping into the system happens while the time derivative of the scalar at the holographic boundary is nonvanishing. The profile (13) is within 3% of its final value at $t/\tau \approx 1.75$. We observe in Fig. 1 that instead j_X equilibrates at $tT \approx 1.2$. Hence, the energy density and the current generated in fast quenches have independent equilibration time scales, τ and $1/T$, respectively. This provides an alternative and simple way to discard a near equilibrium evolution, since no single effective temperature can describe both observables. At the geometrical level the different equilibration time scales have further implications. While the energy density only depends on the bulk total mass, the anomaly free current must be sensitive to the interior geometry close to the emerging horizon. Consistent with this, j_X turns out to mainly build up as equilibrium is approached. The body of Fig. 1(b) describes a very fast quench with $\tau T_0 = 0.007$. The purple curve gives the time profile of the scalar field at the asymptotic boundary (13). The time interval when the quench occurs has been highlighted in blue. Notably, the current is practically zero in this example even sometime after the quench has finished. This clearly shows that the anomaly free current does not react to the initial far from equilibrium state, but to the onset of the equilibrium. A central conclusion of our study is that anomalous transport properties related to the gravitational anomaly are very much linked to the system being at or close to thermal equilibrium.

We wish to now study the transition from fast to slower quenches. Figure 2(a) shows the last stages in the evolution of the current for several processes with the same initial temperature and time span. For the sake of comparison we normalize the final current to 1, and take the span of the quench instead of the final temperature as the time unit. Processes with $2\tau T < 0.5$ behave as explained above. When $2\tau T \approx 0.65$ the maximum before equilibration starts to weaken out, and it practically disappears for $2\tau T \approx 1$. Processes with $2\tau T \gtrsim 1$ exhibit a monotonic growth of the current to its equilibrium value. We will refer to them as intermediate quenches. We have highlighted the time span of the quench, up to the moment when ϕ_0 is within 3% of its final value. The transition between fast and intermediate quenches happens when the time scale set by the final temperature approaches that of the quench. It is important to stress that, in general, intermediate quenches do not admit either a near equilibrium description. Indeed when the final temperature is higher than T_m, the absence of a transient maximum signals out of equilibrium dynamics. For the parameters associated with Fig. 2(a) we have $2\tau T_m = 0.5$, such that processes with final temperature below T_m behave as fast quenches.

Finally, we consider slow quenches. First, we consider processes that are slow with respect to the final temperature and can be expected to have a sizable final period of near-equilibrium evolution. The hydrodynamic current described by the benchmark curve (11) decreases with rising temperature when $T > T_m$. In this case, a final period of near-equilibrium evolution implies that j_X must exhibit a transient maximum. In Fig. 2(b) we investigate quenches with $2\tau T > 2$ for the same initial conditions as in Fig. 2(a). The red curve coincides in both plots for the sake of comparison. We observe that a maximum reappears with a distinctive characteristic different from that exhibited by fast quenches. Unlike them, the maximum of the quotient $j_X/j_{X\text{eq}}$ increases with temperature, and can be attained even before the quench is halfway through. It is natural that the larger τT is the earlier the system enters the near-equilibrium regime, whose onset is qualitatively signaled by the maximum of the current.

In the previous processes $\tau T_0 = 0.175$, such that they are slow with respect to the final temperature but fast with respect to the initial one. This constrains when the near equilibrium regime sets in and hence whether the current can reach 1/4, the maximum of the equilibrium curve (11). Keeping the same τT_0, Fig. 3(a) shows the evolutions with a very high final temperature. Since we are interested in the maximal value of the current, j_X has not been rescaled as we did in Fig. 2. Instead of approaching 1/4, the maximum turns out to slowly decrease with increasing final temperature. This shows that the near-equilibrium regime in processes with $\tau T_0 = 0.175$ can only be associated with temperatures well above T_m. In order for the complete

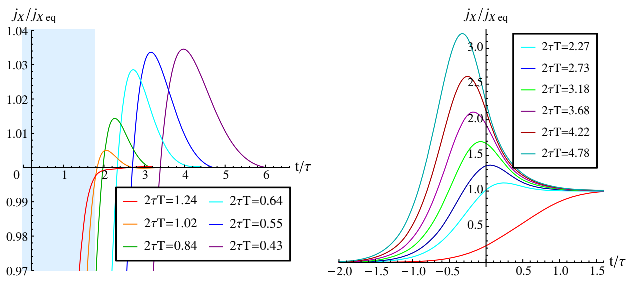

FIG. 2. Left: final approach to equilibrium of j_X for several quenches with $\tau T_0 = 0.175$. Right: evolution of j_X in processes with $\tau T_0 = 0.175$ and a high final temperature. The red curve coincides on both plots for comparison.

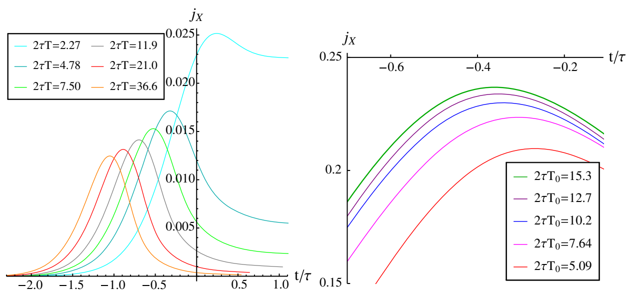

FIG. 3. Left: evolution of j_X in processes with $\tau T_0 = 0.175$ and a very high final temperature. Right: processes with fixed initial and final temperatures $T/T_0 = 2.6$ and growing τ.

evolution to be hydrodynamic, the quench also needs to be slow with respect to the initial temperature. We plot in Fig. 3(b) processes with fixed $T/T_0 = 2.6$ and growing time span. The maximum of the current tends now to 1/4 as expected. We have checked that, consistently, the current builds up in a monotonic way for these slow processes whose final temperature is equal or smaller than T_m.

We have studied holographic quantum quenches of the CME induced by the gravitational anomaly. A rich phenomenology arises depending on the time scale of the quench. If the quench is fast with respect to the initial and final temperatures, the evolution is far from equilibrium until the final exponential approach to stabilization. The current builds up late in the time evolution and slightly overshoots before it achieves its equilibrium value. In equilibrium the anomalous conductivity is proportional to the square of the temperature. The main motivation of this work was to analyze what activates the anomalous conductivity out of equilibrium, where there is no notion of temperature. It could have been governed by energy density, which in equilibrium is also measured by the temperature. In this case the current should have reacted as soon as energy is injected into the system. Our result on fast quenches shows that this is not the case. Rather the system has to evolve closer to equilibrium to build up the anomalous current. This is in contrast to the immediate response of other holographic one-point functions, as observed, for example, in Ref. [34].

Intermediate quenches leave the far from equilibrium stage while they are reaching the final state, resulting in a monotonic growth of the current. Processes that are slow with respect to the final temperature but fast with respect to the initial one have a finite period of near-equilibrium evolution. This extends to the complete evolution for large τT_0.

While we have restricted ourselves to the weak field regime appropriate for linear response, it will also be interesting to study the low temperature or high magnetic field regime. Since the buildup of the current requires being close to equilibrium we expect our results to carry over.

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3

SUPERTRANSLATIONS: REDUNDANCIES OF HORIZON DATA AND GLOBAL SYMMETRIES AT NULL INFINITY

This chapter contains the article

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Supertranslations: redundancies of horizon data and global symmetries at null infinity

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Abstract

We characterise the geometrical nature of smooth supertranslations defined on a generic non-expanding horizon (NEH) embedded in vacuum. To this end we consider the constraints imposed by the vacuum Einstein's equations on the NEH structure, and discuss the transformation properties of their solutions under supertranslations. We present a freely specifiable data set which is both necessary and sufficient to reconstruct the full horizon geometry, and is composed of objects which are invariant under supertranslations. We conclude that smooth supertranslations do not transform the geometry of the NEH and that they should be regarded as pure gauge. Our results apply both to stationary and non-stationary states of a NEH, the latter ones being able to describe radiative processes taking place on the horizon. As a consistency check we repeat the analysis for Bondi–Metzner–Sachs (BMS) supertranslations defined on null infinity, $\mathcal{I}$. Using the same framework as for the NEH we recover the well-known result that BMS supertranslations act non-trivially on the free data on $\mathcal{I}$. The full analysis is made in exact, non-linear, general relativity.

Keywords: black holes, horizon, asymptotic symmetries, supertranslations, BMS group, null infinity

1. Introduction

The subject of asymptotic symmetries in gravitational theories has been an active field of research in recent years. One of the main motivations to study the asymptotic structure of the spacetime boundary is the characterisation of candidate theories of quantum gravity. A prominent example is the result in [1, 2], that any consistent theory of quantum gravity on a spacetime which is asymptotically AdS_3 should be a conformal field theory. This result has led to major developments in the understanding of the microscopic origin of black hole entropy, as in the case of the Bañados–Teitelboim–Zanelli (BTZ) black hole [3–5], and for extremal Kerr black holes in four dimensions [6]. The success of this approach has inspired many attempts to extend these results to the case of astrophysical—non-extremal—black holes in asymptotically flat spacetimes (see [7–20] and references therein).

The symmetry group of four-dimensional asymptotically flat spacetimes at null infinity $\mathcal{I}$ is the so called Bondi–Metzner–Sachs (BMS) group [21–23]. This group consists of the semidirect product of the Lorentz group times an infinite dimensional abelian normal subgroup which generalises translations, the so-called *supertranslations*. Supertranslations act on future (past) null infinity by shifting the advanced (retarded) time independently for each point of the sphere at infinity. These diffeomorphisms are particularly interesting because they act non-trivially on the geometric data at null infinity, which encodes the gravitational degrees of freedom of gravitational radiation. More specifically, the radiative vacua of asymptotically flat spacetimes is infinitely degenerate, and supertranslations act transitively on it, i.e. all radiative vacua are connected to each other by supertranslations. The implications of this symmetry group on the gravitational S-matrix were first studied in the framework of *asymptotic quantization* [24–26] (see also [27, 28]), and the relation between supertranslations and Weinberg's *soft graviton theorem* [29] has been explored in [30–34].

Recently, it has been argued that in black hole spacetimes, if the event horizon is regarded as an inner boundary, it is appropriate to enhance the asymptotic symmetry group (ASG) with those diffeomorphisms which leave invariant the near horizon geometry[4] [8, 10, 14, 15, 20, 33–42]. For stationary black holes the corresponding set of diffeomorphisms has been shown to be a reminiscence of the BMS group on $\mathcal{I}$. As in the case of null infinity, the ASG on the horizon is enhanced with respect to the isometry group of the background with the addition of supertranslations, which in this case shift the advance (retarded) time of the future (past) horizon. Following the analogy with the BMS group on null infinity, it has been conjectured that the horizon supertranslations may act non-trivially on the black hole geometry, transforming the black hole to a physically inequivalent one. If this was the case, the 'supertranslation hair' could provide some insight on the microscopic degrees of freedom associated to the black hole entropy. Although these ideas are certainly appealing, the physical nature of the asymptotic symmetry group defined on non-extremal horizons is still unclear, and the proposal remains controversial [43–45].

On the one hand, there are still some discrepancies on the structure of the ASG found in different analyses [8, 10, 14, 15, 20, 36, 46]. These differences could be attributed to alternative choices of boundary conditions for the metric tensor near the horizon, but the geometric interpretation of the discrepancies is not well understood. The main difficulty in comparing different analyses is that they are based on a coordinate dependent approach. In practice, the boundary conditions are defined as restrictions on an explicit coordinate expression of the metric tensor in the neighbourhood of the horizon. This complicates the comparison between the various works as, in general, a given set of boundary conditions does not retain the same form in different coordinate systems.

On the other hand, it remains an open question to determine the effect of the horizon ASG on the geometry, and in particular, whether these diffeomorphisms act non-trivially on the physical state of the black hole. This problem was recently addressed in [34], where the authors performed a Hamiltonian analysis to characterise the phase space of a Schwarzschild spacetime. According to this study the phase space of the Schwarzschild spacetime is infinite

[4] A more complete list of related works can be found in [34].

dimensional, and supertranslations act non-trivially on it. This conclusion contrasts with the classical result that there is only a three-parameter family of stationary black hole solutions in Einstein–Maxwell theory. Actually, the Hamiltonian analysis of spacetimes containing isolated horizons, such as the Schwarzschild spacetime, had been considered before in [47–50]. In those works it was shown that the corresponding phase space could be infinite dimensional in the presence of radiation, but in the stationary case it was argued that the physical state is completely determined by the standard quantities: ADM mass, angular momentum and electric charge.

In the present paper we will study horizon supertranslations defined on a generic non-expanding horizon (NEH) which is embedded in vacuum [47–51]. A non-expanding horizon is a generalisation of a killing horizon which admits gravitational radiation propagating arbitrarily close to it, and even *on* the NEH itself (but not crossing it). Our main objective is to provide a coordinate invariant definition of horizon supertranslations, and then to characterise their effect on the NEH geometry. For this purpose we have taken as a guide the geometric method used by Geroch [52] and Ashtekar [24–26, 53, 54] to study the structure and dynamics of null infinity. Following these works, we describe the horizon in terms of an abstract three-dimensional manifold separated from the spacetime, and which is diffeomorphically identified with the horizon. In this framework, the information about the intrinsic and extrinsic geometry of the horizon is encoded in tensor fields living on the abstract manifold. The advantage of this method is that the geometric data of the horizon is isolated from the rest of the spacetime, and moreover, all the gauge redundancies are well characterised.

To clarify the geometrical nature of supertranslations we have studied the set spacetime diffeomorphisms which preserve the horizon as a set of points, and leave invariant the metric tensor on it. Note that these diffeomorphisms, which we call for short *hypersurface symmetries*, are defined in a coordinate invariant way, i.e. without involving an explicit coordinate expression for the spacetime metric tensor. After checking that horizon supertranslations belong to this class of diffeomorphisms, we have studied the behaviour of the complete horizon geometry (including the extrinsic geometry) under an arbitrary hypersurface symmetry. Our analysis shows that the effect of these diffeomorphisms on the horizon can be identified with a gauge redundancy of the description. In other words, hypersurface symmetries, and in particular supertranslations, leave invariant both *the intrinsic and the extrinsic* geometry of the horizon up to a gauge redundancy of the description. Note, however, that this result is not sufficient to claim that supertranslations act trivially on the geometry of the horizon. Indeed, supertranslations could be large gauge transformations, i.e. *global symmetries*, which can change the dynamical state of a system [1, 2, 55]. Thus, we still need to identify the dynamical degrees of freedom of the horizon, or equivalently, a data set which is both *necessary and sufficient* to reconstruct the full NEH geometry, and then we have to determine how it transforms under supertranslations.

In order to identify the dynamical degrees of freedom of the NEH we follow the geometric method of [56] and [50] (see also [57, 58]), which consists of studying the constraints imposed by the vacuum Einstein's equations on its geometric data. By solving these constraint equations it is possible to extract a set of freely specifiable quantities which contain all the information necessary to reconstruct the NEH geometry [50]. Thus, the resulting *free horizon data set* encodes the dynamical degrees of freedom of the horizon, but in general it also involves some gauge redundancies. In the present work we have reconsidered the analysis in [50] for non-expanding horizons, discussing in detail the treatment of the gauge redundancies, and specifically of supertranslations.

The main result of this work is the identification of a free data set which does not involve any unfixed gauge degree of freedom and, in particular, which is composed of objects which are *invariant under supertranslations*. An immediate consequence of our result is that supertranslations do not affect the NEH geometry, as it can be encoded entirely in quantities which are invariant under these diffeomorphisms. In particular, the stationary state of the horizon is completely determined by its intrinsic geometry and its angular momentum aspect, and neither of the two transform under horizon supertranslations. The supertranslation invariant data set is also sufficiently general to represent non-stationary states of the NEH, and thus, it can describe radiative processes occurring at the horizon. It is important to stress that, to avoid excluding physically allowed configurations of the horizon, we have not eliminated the freedom to perform supertranslations using gauge fixing conditions. Instead, guided by the treatment of null infinity [53], we have dealt with the redundancy describing the NEH geometry in terms of variables which are invariant under supertranslations. Our conclusions are directly applicable to the non-singular horizon supertranslations discussed in [8, 15, 20, 35, 36, 39, 41, 46]. The supertranslations studied in [34] cannot be described as hypersurface symmetries, and thus we will consider this case in a separate publication [59].

As a consistency check we have repeated the analysis of null infinity as in [53] using the same framework as for non-expanding horizons. In particular, following [53], we have characterised the solutions to the constraint equations of null infinity in terms of variables invariant under *BMS supertranslations*, and we have reproduced the proof of the degeneracy of the radiative vacuum of asymptotically flat spacetimes. In other words, we find that the solution space of the constraint equations of $\mathcal{I}$ in the absence of radiation is infinite dimensional. Since the analysis is made in terms of variables which are free of any gauge redundancies, these degenerate vacua must be regarded as physically distinct, and yet they can be shown to be connected to each other by supertranslations. Therefore, we recover the well known result that BMS supertranslations—contrary to the case of horizons—act non-trivially on the free data of null infinity, i.e. they represent a *global symmetry* of the constraint equations.

This article is organised as follows. In section 2 we review the formalism to describe the geometry of null hypersurfaces, together with the constraint equations that restrict the corresponding geometric data. In section 3 we characterise in detail the gauge redundancies inherent to our description, and we discuss the effect of supertranslations on the horizon geometry. In section 4 we analyse the constraint equations of a non-expanding horizon, and present a free data set to describe its geometry which is composed of quantities invariant under supertranslations. In section 5 we consider the constraint equations for null infinity, and we reproduce the proof of the degeneracy of the radiative vacuum of asymptotically flat spacetimes using our framework. Finally in section 6 we discuss our results.

2. Dynamics of null hypersurfaces

In this section we will review the geometry and dynamics of null hypersurfaces, what will also serve to present the relevant formulae. A more detailed overview of this subject can be found in [58, 60], while the specific framework used here is based on [57].

We begin setting our notation and general conventions. We will work with $(3+1)$−dimensional spacetimes $(\mathcal{M}, g)$, described by a manifold $\mathcal{M}$ equipped with a metric tensor g with signature $(-,+,+,+)$. We will denote the spacetime coordinates by $\{x^\mu\}$, with the index running over $\mu = 0, 1, 2, 3$. The Riemann tensor is defined in terms of the Ricci identity as follows

$$\nabla_{[\mu}\nabla_{\nu]}V^\sigma \equiv \nabla_\mu\nabla_\nu V^\sigma - \nabla_\nu\nabla_\mu V^\sigma = R^\sigma{}_{\rho\mu\nu}V^\rho, \tag{2.1}$$

where V^μ is an arbitrary vector field[5], the Ricci tensor is given by $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$, and the scalar curvature by $R = R^\mu_\mu$. The Riemann curvature can be split in its trace part, characterised by the Schouten tensor $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{6}Rg_{\mu\nu}$, and its traceless part, encoded in the Weyl tensor $C_{\mu\nu\rho\sigma}$

$$R_{\sigma\rho\mu\nu} = C_{\sigma\rho\mu\nu} + \frac{1}{2}(g_{\sigma[\mu}S_{\nu]\rho} - g_{\rho[\mu}S_{\nu]\sigma}). \tag{2.2}$$

We will use geometrized units $c = G = 1$ so that the Einstein's equations read

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{2.3}$$

In regions where the spacetime geometry is consistent with the vacuum Einstein's equations, $R_{\mu\nu} = 0$, the Schouten tensor must be zero and thus the Riemann curvature is completely determined by the Weyl tensor $R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}$.

2.1. Geometric data of null hypersurfaces

In this section we will review the geometry and dynamics of a null hypersurface $\mathcal{H}$. Bearing in mind the case of black hole horizons and null infinity, we will assume the hypersurface to have the topology $\mathcal{H} \cong \mathbb{R} \times \mathbb{S}^2$. We will describe the hypersurface as the embedding of an abstract three dimensional manifold $\Sigma \cong \mathbb{R} \times \mathbb{S}^2$ on the spacetime via the diffeomorphism $\Phi : \Sigma \to \mathcal{M}$, so that $\Phi(\Sigma) = \mathcal{H}$ (see [57]). The manifold Σ acts as a diffeomorphic copy of $\mathcal{H}$ detached from the spacetime, and it is introduced for convenience in order to isolate the dynamical degrees of freedom (i.e. the free geometric data) of the hypersurface.

To characterise the intrinsic and extrinsic geometry of the hypersurface it is convenient to introduce a basis of the spacetime tangent space adapted to $\mathcal{H}$. For this purpose let us first define a coordinate system for the abstract manifold $\{\xi^a\}$, with the index running over $a = 1, 2, 3$. The corresponding coordinate basis of the tangent space $T_p\Sigma$ is then given by $\hat{\mathcal{B}} \equiv \{\hat{e}_a = \partial_{\xi^a}\}$, with $p \in \Sigma$. The elements of the basis $\hat{\mathcal{B}}$ are identified with a set of linearly independent spacetime vectors tangent to the hypersurface $e_a \equiv \mathrm{d}\Phi(\hat{e}_a)$, via the pushforward map $\mathrm{d}\Phi$ associated to Φ. Then, we can form a basis $\mathcal{B} = \{e_a, \ell\}$ of the spacetime tangent space completing the set of vectors $\{e_a\}$ with any vector ℓ transverse to $\mathcal{H}$, the so called *rigging*.

The spacetime metric over the hypersurface can be characterised in terms of a set of tensor fields over Σ which, by definition, have the following components on the basis $\hat{\mathcal{B}}$

$$\gamma_{ab} \equiv g(e_a, e_b)|_{\Phi(p)}, \qquad \ell_a \equiv g(\ell, e_a)|_{\Phi(p)}, \qquad \ell^{(2)} \equiv g(\ell, \ell)|_{\Phi(p)}. \tag{2.4}$$

These fields encode the scalar products of the elements in the spacetime basis $\mathcal{B} = \{e_a, \ell\}$, and in particular γ_{ab} represents the induced metric on $\mathcal{H}$. This set of fields is known as the *hypersurface metric data*.

In order to reduce the large degree of gauge freedom in this description, namely the choice of coordinates on Σ and the specification of the rigging vector ℓ, it is useful to introduce some simplifying conventions. The normal one-form $\boldsymbol{n}$ and the normal vector n to the hypersurface are determined by the conditions $\boldsymbol{n}(e_a) = 0$ and $n = g^{-1}(\boldsymbol{n}, \cdot)$, respectively, and thus they are defined up to $\boldsymbol{n} \to \lambda\boldsymbol{n}$ and $n \to \lambda n$, where λ is a scalar field on $\mathcal{H}$. Since the normal vector to a null hypersurface is null, i.e. $g(n, n) = \boldsymbol{n}(n) = 0$, n is also tangent to $\mathcal{H}$, and we will choose it to be future directed. Therefore we can partially fix the coordinate system on the abstract

[5] We will use the shorthand $W_{[\mu\nu]} = W_{\mu\nu} - W_{\nu\mu}$ and $W_{(\mu\nu)} = W_{\mu\nu} + W_{\nu\mu}$ to denote the symmetrisation and anti-symmetrisation of indices.

manifold Σ defining ξ^1 so that $e_1 = n$, and parametrising the $\mathbb{S}^2$ component of the hypersurface with the coordinates ξ^M, where $M = \{2,3\}$. Moreover we will require the rigging vector ℓ to be null $g(\ell,\ell) = 0$, and we will fix its normalisation and direction with respect to the vectors $\{e_a\}$ imposing the conditions $\boldsymbol{n}(\ell) = 1$ and $g(\ell, e_M) = 0$ everywhere on $\mathcal{H}$, what can be expressed equivalently as $g(\ell, e_a) = \delta^1_a$. With these choices the explicit coordinate expressions for the hypersurface metric data in the basis $\hat{\mathcal{B}} = \{\hat{e}_1, \hat{e}_M\}$ read

$$\gamma_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & q_{MN} \end{pmatrix}, \qquad \ell_a = (1,0,0), \qquad \ell^{(2)} = 0. \tag{2.5}$$

Here $q_{MN} \equiv g(e_M, e_N)|_{\Phi(p)}$ represents the induced metric on the spatial sections $\mathcal{S}_{\xi^1} \equiv \Sigma|_{\xi^1}$ of the horizon, which are defined by the level sets of the null parameter ξ^1. In the following we will denote the elements of the basis of the spacetime tangent space by $\mathcal{B} = \{n, \ell, e_M\}$. For later convenience, we also write here the following identity satisfied by the elements of $\mathcal{B}$

$$g^{\mu\nu} = \ell^{(\mu} n^{\nu)} + q^{MN} e^{\mu}_N e^{\nu}_M. \tag{2.6}$$

The Levi-Civita connection at points of the hypersurface can be characterised specifying its action on the elements of the basis $\mathcal{B}$

$$\begin{aligned} &\nabla_n n = \kappa n, && \nabla_M n = \Omega_M n + \Theta_M{}^N e_N, \\ &\nabla_n e_M = \Omega_M n + \Theta_M{}^N e_N, && \nabla_M e_N = -\Theta_{MN}\ell - \Xi_{MN} n + \overline{\Gamma}^L_{MN} e_L, \\ &\nabla_n \ell = -\kappa\ell - \Omega^M e_M, && \nabla_M \ell = -\Omega_M \ell + \Xi_M{}^N e_N, \end{aligned} \tag{2.7}$$

where the indices M,N are raised and lowered with q_{MN} and its inverse q^{MN}, $\nabla_n \equiv n^\mu \nabla_\mu$ and $\nabla_M \equiv e^\mu_M \nabla_\mu$. This is the most general form of the connection coefficients consistent with our conventions (2.5), as it can be easily derived in the framework of [57]. In particular, it can be seen that the integral curves of the normal vector n are null geodesics parallel to the hypersurface, and the inaffinity parameter κ is referred as the *surface gravity* in the case of horizons. The surface gravity, together with the *Hajicek one-form* Ω_M and Ξ_{MN} can be conveniently encoded in the tensor field Y_{ab} defined in terms of the basis $\hat{\mathcal{B}}$ of $T_p\Sigma$ as follows

$$Y_{ab} \equiv \frac{1}{2}\mathcal{L}_\ell g(e_a, e_b)|_{\Phi(p)} = \begin{pmatrix} -\kappa & -\Omega_M \\ -\Omega_N & \Xi_{MN} \end{pmatrix}. \tag{2.8}$$

The set of coefficients $\Xi_{MN} = \frac{1}{2} e^\mu_M e^\nu_N \nabla_{(\mu}\ell_{\nu)}$ characterises the components of the Levi-Civita connection associated to directions which are all transverse to the normal vector n, and thus we will refer to it as the *transverse connection*. For later convenience we will also introduce the rotation one-form ω_a which is defined by

$$\omega_a \equiv -Y_{ab}\,\hat{n}^b = (\kappa, \Omega_M), \tag{2.9}$$

where $\hat{n}$ is the vector on $T_p\Sigma$ which is identified with the null normal via the embedding $\mathrm{d}\Phi(\hat{n}) \equiv n$. The remaining connection coefficients, Θ_{MN} and $\overline{\Gamma}^C_{BA}$, are fully determined by the intrinsic geometry via the equations

$$\frac{1}{2}\partial_n q_{MN} = \Theta_{MN}, \qquad \overline{\Gamma}^L_{MN} = \frac{1}{2} q^{LP}(\partial_M q_{NP} + \partial_N q_{MP} - \partial_P q_{MN}), \tag{2.10}$$

where $\partial_n q \equiv \partial_{\xi^1} q$. Thus, $\overline{\Gamma}^C_{BA}$ represents the Levi-Civita connection compatible with q_{MN}, and the quantity Θ_{MN} is known as the *second fundamental form*.

Summarising, the intrinsic and extrinsic geometry of a null hypersurface can be fully encoded in the following set of fields defined on Σ

$$\boxed{\text{Hypersurface data}: \qquad \mathscr{D} \equiv (q_{MN}, \quad \kappa, \quad \Omega_M, \quad \Xi_{MN}).} \tag{2.11}$$

Our choice of coordinates for $\Sigma \cong \mathbb{R} \times \mathbb{S}^2$, with ξ^1 running along the null direction n of the hypersurface, and ξ^M parametrising the sections with constant ξ^1, $\mathcal{S}_{\xi^1} \cong \mathbb{S}^2$, allows one to picture the data κ, Ω_M and Ξ_{MN} as tensor fields living on a manifold with the topology of a sphere and a Riemannian metric q_{MN}. In this picture, the dependence of these fields on the null coordinate ξ^1 is interpreted as a 'temporal' evolution [61–63]. As we shall see in the next section, the evolution of these fields along the null direction is not completely free, as it is restricted by the geometry of the ambient space. Moreover, as we shall see in sections 3 and 4, this description still involves some residual gauge redundancies, which lead to further constraints on (2.11) after gauge fixing.

In the following we will often identify the abstract manifold Σ with the hypersurface $\mathcal{H}$, and we will leave implicit the pull-back Φ^* operation in the formulae to simplify the notation.

2.2. Constraint equations of null hypersurfaces

The spacetime connection on the hypersurface, given by (2.7), must be consistent with the geometry of the ambient space where $\mathcal{H}$ is embedded. This requirement leads to the *hypersurface constraint equations* which relate the connection coefficients in (2.7) with certain projections of the Ricci tensor $R_{\mu\nu}$ at points of the hypersurface $\mathcal{H}$. When expressed in terms of the hypersurface data $\mathscr{D}$ (2.11) these mathematical identities take the form of a set of equations of motion, which we will now review. A formal analysis of these equations can be found in [57, 64], and their application to null hypersurfaces is reviewed in detail in [58]. Since our conventions do not match the ones in these references, for completeness we have included a derivation of these formulae in appendix A. The relevant equations are:

- *Raychaudhuri equation*:

$$\boxed{\partial_n\theta - \theta\kappa + \Theta_{MN}\Theta^{MN} = J_{nn}.} \tag{2.12}$$

- *Damour–Navier–Stokes equations*:

$$\boxed{\partial_n\Omega_M - \partial_M\kappa + \theta\Omega_M + D_N\Theta^N_M - D_M\theta = -J_{nM}.} \tag{2.13}$$

- Equation for the *transverse connection*:

$$\boxed{\begin{aligned} \partial_n\Xi_{MN} = \; & -\tfrac{1}{2}D_{(M}\Omega_{N)} - \Omega_M\Omega_N - (\kappa + \tfrac{1}{2}\theta)\Xi_{MN} + \Xi_{P(M}\Theta^P_{N)} \\ & -\tfrac{1}{2}\Theta_{MN}\theta^\ell + \tfrac{1}{4}\mathcal{R}q_{MN} + \tfrac{1}{2}J_{MN}. \end{aligned}} \tag{2.14}$$

Here $\theta \equiv \Theta^M_M$ is the *expansion* of the null hypersurface, and $\theta^\ell \equiv \Xi^M_M$. The symbols D_M and $\mathcal{R}$ denote the Levi-Civita connection of q_{MN} and the associated Ricci scalar, respectively. The equations also involve the tensor J_{ab} defined on Σ in terms of its components in the basis $\hat{\mathcal{B}} = \{\hat{e}_a\}$

$$J_{nn} \equiv -\mathbf{R}(n,n)|_{\Phi(p)}, \quad J_{nM} \equiv -\mathbf{R}(n,e_M)|_{\Phi(p)}, \quad J_{MN} \equiv -\mathbf{R}(e_M,e_N)|_{\Phi(p)}, \tag{2.15}$$

where $\mathbf{R}(e_a, e_b) = R_{\mu\nu} e_a^\mu e_b^\nu$ represent projections of the spacetime Ricci tensor. The constraint equations (2.12)–(2.14) simplify considerably in the particular case of *non-expanding horizons* which are embedded in vacuum. If the Ricci tensor $R_{\mu\nu}$ is consistent with the Einstein's field equations (2.3), the quantities (2.15) can be associated to projections of the energy–momentum tensor on the basis $\mathcal{B}$. Therefore they are all vanishing $J_{nn} = J_{nM} = J_{MN} = 0$ in vacuum $T_{\mu\nu} = 0$. By definition, a non-expanding horizon is a null hypersurface which has a vanishing expansion θ [47, 50] (see also [58]), and then, due to the vacuum Raychaudhuri equation (2.12), we must have

$$\boxed{\text{Non-expanding horizon}: \quad \theta = 0 \quad \Longrightarrow \quad \frac{1}{2}\partial_n q_{MN} = \Theta_{MN} = 0.} \tag{2.16}$$

As a consequence, the spatial metric q_{MN} induced on the sections of a non-expanding horizon $\mathcal{S}_{\xi^1}$ is independent on the null coordinate ξ^1. Moreover, for a NEH the equations (2.13) and (2.14) reduce to

$$\partial_n \Omega_M = \partial_M \kappa, \tag{2.17}$$

$$\partial_n \Xi_{MN} = -\frac{1}{2} D_{(M}\Omega_{N)} - \kappa \Xi_{MN} - \Omega_M \Omega_N + \frac{1}{4} q_{MN} \mathcal{R}, \tag{2.18}$$

where we have already imposed the vacuum Einstein's equations.

As we shall review in section 5, null infinity $\mathcal{I}$ can be described as an non-expanding hypersurface with $\Theta_{MN} = 0$ using Penrose's conformal framework, and its structure is also constrained by (2.12)–(2.14), which are mathematical identities satisfied by any null hypersurface. However, non-expanding horizons and null infinity have very different dynamical behaviour, and in particular, the constraints (2.17) and (2.18) are not valid for $\mathcal{I}$. One of the reasons is that the geometric data of $\mathcal{I}$ is only defined up to conformal transformations, what requires introducing appropriate equivalence classes of data sets [53]. The other important difference with NEHs is that the Ricci tensor defined on the conformal completion of spacetime does not satisfy the ordinary Einstein's equations, and thus a specific treatment is required for $\mathcal{I}$.

2.3. Newman–Penrose null tetrad and Weyl scalars

The set of constraint equations (2.12)–(2.14), ensures the consistency of the connection coefficients in (2.7) with the trace part of the ambient-space Riemann tensor, i.e. the Ricci tensor $R_{\mu\nu}$. Therefore, it is possible to obtain further constraints requiring that the extrinsic geometry of $\mathcal{H}$ to be compatible with the traceless part of the curvature, that is, with the Weyl tensor $C_{\mu\nu\rho\sigma}$. The Weyl tensor has 10 independent components which can be collected in the form of five independent complex scalars Ψ_n, with $n = 0, \ldots, 4$, the so called *Weyl scalars*.

In order to define the Weyl scalars, first we have to introduce a *Newman–Penrose null tetrad* (see e.g. [65, 66]), what can be done in our framework as follows. At any given point ξ_0^M of the spatial sections $\mathcal{S}_{\xi^1}$ it is possible to find a set of coordinates ξ^M such that the spatial metric q_{MN} has the simple form $q_{MN}(\xi_0) = \delta_{MN}$. Note that for NEH horizons this choice is independent of ξ^1 as $\partial_n q_{MN} = 0$. In this way we ensure that the two basis vectors $e_M|_{\xi_0^M}$ are orthogonal to each other and have unit norm. Then, we can construct the Newman–Penrose null tetrad $\mathcal{B}_{NP} = \{n, \ell, m, \overline{m}\}$ at $\{\xi^1, \xi_0^M\}$ comprised of the normal vector n, the rigging ℓ, and the two complex null vectors

$$m|_{\xi_0^M} \equiv \frac{1}{\sqrt{2}}(e_2 + \mathrm{i} e_3)|_{\xi_0^M}, \qquad \overline{m}|_{\xi_0^M} \equiv \frac{1}{\sqrt{2}}(e_2 - \mathrm{i} e_3)|_{\xi_0^M}. \tag{2.19}$$

By defining the $\mathcal{B}_{NP}$ in this way we avoid introducing the additional gauge freedom which is always associated to the choice of null tetrad. The set of vectors $\mathcal{B}_{NP}$ also forms a basis of the spacetime tangent space, and it is composed of null vectors only. Actually, at ξ_0^M the scalar products of its elements read

$$\begin{aligned} &g(n,n)=0, \quad g(n,\ell)=1, \quad g(n,m)=0, \quad g(n,\overline{m})=0,\\ &g(\ell,\ell)=0, \quad g(\ell,m)=0, \quad g(\ell,\overline{m})=0,\\ &g(m,m)=0, \quad g(m,\overline{m})=1,\\ &g(\overline{m},\overline{m})=0. \end{aligned} \tag{2.20}$$

The Weyl scalars are defined in terms of the Newman–Penrose tetrad by[6]

$$\begin{aligned} &\Psi_0 = C_{\sigma\rho\mu\nu}n^\sigma m^\rho n^\mu m^\nu, \qquad \Psi_1 = C_{\sigma\rho\mu\nu}n^\sigma m^\rho \ell^\mu n^\nu,\\ &\Psi_2 = C_{\sigma\rho\mu\nu}n^\sigma m^\rho \ell^\mu \overline{m}^\nu, \qquad \Psi_3 = C_{\sigma\rho\mu\nu}n^\sigma \ell^\rho \overline{m}^\mu \ell^\nu,\\ &\Psi_4 = C_{\sigma\rho\mu\nu}\overline{m}^\sigma \ell^\rho \overline{m}^\mu \ell^\nu. \end{aligned} \tag{2.21}$$

The computation of the Weyl scalars is useful to determine the Petrov type of the gravitational field (see [66]), and to characterise its different contributions [67]. In particular, Ψ_0 and Ψ_4 encode transverse wave components travelling along the directions[7] $-\ell$ and n, respectively. The scalars Ψ_1 and Ψ_3 represent longitudinal wave components propagating respectively parallel to $-\ell$ and n, and Ψ_2 can be associated with a Coulomb contribution of the gravitational field [67].

The general form of the Weyl scalars in terms of the hypersurface data (2.11) can be found appendix B.2. We will present the relevant formulae when discussing case of non-expanding horizons and null infinity.

3. Gauge redundancies and horizon supertranslations

In the present section we will discuss in detail the gauge redundancies in our description in the case of generic non-expanding null hypersurfaces with $\Theta_{MN}=0$, which are of interest both for the study of black hole horizons and null infinity. Part of this gauge freedom was already used in the last section to set the hypersurface metric data in the form (2.5). Then, in section 3.2 we will begin our analysis identifying the residual gauge redundancies which are left after imposing the conventions (2.5). Since the constraints (2.12)–(2.14) are a direct consequence of the Ricci identity and (2.5), these residual gauge transformations also leave invariant the form of the constraint equations.

In section 3.3, we will turn our attention to horizon supertranslations. We will define them as spacetime diffeomorphisms preserving the metric tensor on a generic NEH. We will characterise how the geometry of the horizon changes under the action of a supertranslation, and show that the transformation of the hypersurface data (2.11) can be identified with a gauge redundancy of the description. In other words, we prove that horizon supertranslations preserve both the intrinsic and extrinsic geometry of the horizon up to a gauge transformation. As a consequence, we find that supertranslations also leave invariant the form of the constraint equations (2.12)–(2.14).

[6] We use the definitions in [58] up to a difference in the conventions: in [58] the second element of the tetrad $\mathcal{B}_{NP}$ is future directed, but in our conventions the rigging ℓ is past directed.
[7] The vector $-\ell$ is future directed since, by convention, $g(n,\ell)=1$.

It is important to stress that, the fact that two data sets can be related to each other by a gauge transformation is *necessary* for them to describe *the same NEH geometry*. However, the gauge equivalence of two data sets is *not sufficient* to prove that they correspond to the same NEH. For this reason the results derived in this section do not yet prove that two data sets related by a horizon supertranslation represent the same horizon geometry. Although this might seem unnatural at first, recall that in the case of null infinity there are geometrically distinct data sets which can be related to each other by a gauge transformation, namely by BMS supertranslations. In this sense, BMS supertranslations should be regarded as a large gauge transformation, i.e. a global symmetry of the constraint equations, rather than pure gauge. We will discuss this point again in section 5.

3.1. Gauge redundancies of hypersurface data

As we described in section 2.1, the intrinsic and extrinsic geometry of a null hypersurface can be fully encoded in a set of tensor fields defined on the abstract manifold Σ, namely $\mathscr{D} = (\gamma_{ab}, \ell_a, \ell^{(2)}, Y_{ab})$ defined in (2.4) and (2.8) [68]. The description of the geometry of a non-expanding horizon in terms of these quantities has some 'built in' gauge redundancies, that is, different data sets $\mathscr{D}$ and $\mathscr{D}'$ might represent equivalent geometries. The following two types of redundancies represent *all the gauge ambiguities* in our framework:

3.1.1. Coordinate freedom on the abstract manifold. We recall that the hypersurface data $\mathscr{D}$ is given in terms of tensor fields living on the abstract manifold Σ, and thus its definition is unaffected by coordinate reparametrisations of Σ. Nevertheless, the explicit coordinate expressions of these tensor fields will have a different form in different coordinate systems. This implies that the same hypersurface geometry could be encoded in two different data sets $\mathscr{D}$ and $\mathscr{D}'$ which are related to each other through a *diffeomorphism of the abstract manifold* Σ. These transformations should be regarded as a gauge freedom in the hypersurface data[8]. Under an arbitrary diffeomorphism on the abstract manifold Σ, with the explicit form $\zeta : \xi^a \to \zeta^i(\xi^a)$ and $i = 1, 2, 3$, the coordinate representation of the data $\mathscr{D} = (\gamma_{ab}, \ell_a, \ell^{(2)}, Y_{ab})$ transforms as follows

$$\gamma'_{ab} = \zeta^*\gamma_{ab}, \qquad \ell'_a = \zeta^*\ell_a, \qquad \ell^{(2)\prime} = \zeta^*\ell^{(2)}, \qquad Y'_{ab} = \zeta^*Y_{ab}. \tag{3.1}$$

Here ζ^* is the pull-back map associated to ζ which, for example, acts explicitly on the first fundamental form as $\zeta^*\gamma_{ab} = \gamma_{ij}|_{\zeta(\xi)}\,\zeta^i_a\zeta^j_b$, with $\zeta^i_a \equiv \partial_a\zeta^i$.

At this point it is worth to emphasise that the abstract manifold Σ is a redundant object detached from the ambient spacetime, which can be seen as a bookkeeping device used to isolate the geometric data of the hypersurface $\mathcal{H} \subseteq \mathcal{M}$. Therefore the diffeomorphisms on Σ should not be confused with the diffeomorphisms of the ambient spacetime $\mathcal{M}$.

3.1.2. Choice of rigging. The rigging ℓ is an auxiliary vector field over $\mathcal{H}$ introduced to specify a direction transversal to the hypersurface. Given a null hypersurface $\mathcal{H}$ we could construct the hypersurface data using two different choices of rigging, leading in general to two different results $\mathscr{D}$ and $\mathscr{D}'$ which obviously represent the same geometry. To characterise the effect of an arbitrary change of rigging, consider two different choices, ℓ and ℓ', related to each other by $\ell' = u(\ell + V)$, where u is a non-vanishing scalar function on the hypersurface $\mathcal{H}$, and V is

[8] For a detailed discussion on this type of gauge redundancies in the context of perturbation theory in general relativity see e.g. [68].

some vector field tangent to it. Then, from the definition of the hypersurface data it follows that the elements characterising the metric tensor transform as [57]

$$\gamma'_{ab} = \gamma_{ab}, \quad \ell'_a = \hat{u}(\ell_a + \hat{V}^b \gamma_{ab}), \quad \ell^{(2)\prime} = \hat{u}^2(\ell^{(2)} + 2\hat{V}^a \ell_a + \gamma_{ab}\hat{V}^a\hat{V}^b) \tag{3.2}$$

where $\hat{u}$ is a function in Σ defined by $\hat{u} \equiv \Phi^*(u)$ and $\hat{V} \in T_p\Sigma$ is defined by the condition $\mathrm{d}\Phi(\hat{V}) = V$. The tensor Y_{ab} describing the transverse connection coefficients is sent to

$$Y'_{ab} = \hat{u}Y_{ab} + \frac{1}{2}(\partial_a \hat{u}\,\ell_b + \partial_b \hat{u}\,\ell_a) + \frac{1}{2}\mathcal{L}_{\hat{u}\hat{V}}\gamma_{ab}. \tag{3.3}$$

3.2. Partial gauge fixing and residual gauge redundancies

In section 2.1 we introduced the conventions (2.5) to reduce the elements of the hypersurface data down to $\mathscr{D} = (q_{MN}, \kappa, \Omega_M, \Xi_{MN})$, fixing some of the redundancies described above. In addition, we can specify a particular form for the metric q_{MN} to partially fix the coordinate reparametrisations on the spatial sections of the horizon. However, despite all these conventions there is still some residual gauge freedom. Indeed, we are still allowed to perform a diffeomorphism on Σ (3.1) followed by a change of rigging, (3.2) and (3.3), as long as they preserve our choice of gauge (2.5). Thus, the combined transformations must satisfy

$$\gamma'_{ab} = \zeta^*\gamma_{ab} = \gamma_{ab}, \tag{3.4}$$

$$\ell'_a = \hat{u}(\zeta^*\ell_a + \zeta^*\gamma_{ab}\hat{V}^b) = \delta^1_a, \tag{3.5}$$

$$\ell^{(2)\prime} = \hat{u}^2(\zeta^*\ell^{(2)} + 2\zeta^*\ell_a\hat{V}^a + \zeta^*\gamma_{ab}\hat{V}^a\hat{V}^b) = 0 \tag{3.6}$$

where γ_{mn}, ℓ_n and $\ell^{(2)}$ are given by (2.5). To determine the form of the allowed diffeomorphisms we begin solving the first equation (3.4), which reads explicitly

$$\zeta^i_a\zeta^j_b\gamma_{ij}|_{\zeta(\xi)} = \gamma_{ab}|_\xi \quad \Longleftrightarrow \quad \zeta^I_1\zeta^J_1 q_{IJ}|_{\zeta(\xi)} = 0, \quad \zeta^I_M\zeta^J_N q_{IJ}|_{\zeta(\xi)} = q_{MN}|_\xi, \tag{3.7}$$

where $I, J = 2, 3$. On the one hand, q_{MN} is non-degenerate, and thus the first equation on the right implies that the components of the diffeomorphism $\zeta^i(\xi)$ are constant along the null direction, $\zeta^i_1 = 0$. On the other hand, since we are restricting the analysis to non-expanding horizons $\partial_n q_{MN} = 0$, the previous equations are independent of the null coordinate ξ^1. Therefore the last equation in (3.7) implies that $\zeta^i(\xi^M)$ must define an isometry of the metric $q_{MN}(\xi^M)$, while the component $\zeta^1(\xi) \equiv \hat{f}(\xi)$ of the diffeomorphism can be any arbitrary function on Σ.

Although the diffeomorphism $\zeta^i(\xi)$ leaves γ_{ab} invariant, without a compensating change of rigging, it leads to a non-trivial transformation of the components ℓ_a

$$\zeta^*\ell_a(\xi) = \zeta^i_a\ell_i(\zeta) \quad \Longrightarrow \quad \zeta^*\ell_1 = \partial_n\hat{f} \equiv \hat{f}_n, \qquad \zeta^*\ell_M = \partial_M\hat{f} \equiv \hat{f}_M, \tag{3.8}$$

while $\zeta^*\ell^{(2)} = 0$, and thus $\ell^{(2)}$ is unchanged. The appropriate rigging transformation which compensates this change can be found solving the remaining equations (3.5) and (3.6). The solution for $\hat{u}$ and $\hat{V}^a$ reads

$$\hat{u} = \frac{1}{\hat{f}_n}, \qquad \hat{V}_M = -\hat{f}_M, \qquad \hat{V}^1 = -\frac{1}{2\hat{f}_n}\hat{f}_M\hat{f}^M. \tag{3.9}$$

This result together with the form of the diffeomorphism, $\zeta^i(\xi) = (\hat{f}(\xi), \zeta^I(\xi^M))$, determines completely the residual gauge redundancies of our description. Then, using (3.1) and (3.3) we can find how the form of the tensor Y_{ab} (2.8) changes under this gauge transformations

$$Y'_{ab} = \hat{u}(\zeta^* Y_{ab}) + \frac{1}{2}(\partial_a \hat{u}\,(\zeta^* \ell_b) + \partial_b \hat{u}\,(\zeta^* \ell_a)) + \frac{1}{2}\mathcal{L}_{\hat{u}\hat{V}}\gamma_{ab}. \tag{3.10}$$

Here we have used the fact that the induced metric is invariant under the action of ζ for the last summand (recall (3.4)). Since the freedom to reparametrise the spatial coordinates ξ^M has no interest for our discussion, we consider for simplicity diffeomorphisms with $\zeta^I(\xi^M) = \xi^I$, and we find

$$\boxed{\begin{aligned} \kappa'(\xi) &= \kappa|_{\zeta(\xi)}\hat{f}_n + \partial_n \log \hat{f}_n, \\ \Omega'_M(\xi) &= \Omega_M|_{\zeta(\xi)} + \kappa|_{\zeta(\xi)}\hat{f}_M + \partial_M \log \hat{f}_n, \\ \Xi'_{MN}(\xi) &= \tfrac{1}{\hat{f}_n}\left(\Xi_{MN}|_{\zeta(\xi)} - \Omega_{(M}|_{\zeta(\xi)}\,\hat{f}_{N)} - \kappa|_{\zeta(\xi)}\hat{f}_M\hat{f}_N - D_M\hat{f}_N\right). \end{aligned}} \tag{3.11}$$

Thus, the previous transformations represent the gauge freedom left in the hypersurface data after setting the conventions (2.5). As these transformations leave invariant our conventions (2.5), the form of the constraint equations (2.12)–(2.14) is also left unchanged[9]. This implies that, if a given data set $\mathscr{D}$ is a solution to the constraint equations, then the transformed data set $\mathscr{D}'$ obtained via (3.11) will also satisfy the constraints.

Before we close this discussion let us single out the situation when the diffeomorphism is of the form $\zeta^i(\xi) = (\xi^1 + A(\xi^M), \xi^I)$. Then the gauge transformations have the simpler form

$$\boxed{\begin{aligned} \kappa'(\xi) &= \kappa|_{\zeta(\xi)}, \\ \Omega'_M(\xi) &= \Omega_M|_{\zeta(\xi)} + \kappa|_{\zeta(\xi)}\,A_M, \\ \Xi'_{MN}(\xi) &= \Xi_{MN}|_{\zeta(\xi)} - \Omega_{(M}|_{\zeta(\xi)}\,A_{N)} - \kappa|_{\zeta(\xi)}\,A_M A_N - D_M A_N. \end{aligned}} \tag{3.12}$$

This case is particularly interesting because, as we will see in the next subsection, it is closely related to horizon supertranslations.

Note that a generic change of gauge also induces a transformation on the elements of the basis $\mathcal{B} = \{n, \ell, e_M\}$, and in turn, of the components of any tensor which is expressed in terms of $\mathcal{B}$. The new basis elements $n' = e'_1$ and e'_M at the point ξ^a can be obtained from their definition after acting with the pushforward $\mathrm{d}\zeta$ on $\hat{e}_i$, that is, $e'_m = \mathrm{d}\Phi(\mathrm{d}\zeta(\hat{e}_m)) = \mathrm{d}\Phi(\zeta^i_m \hat{e}_i)$, and the new rigging ℓ' from (3.9). In the case of transformations of the form (3.12) the basis elements behave as

$$n' = n|_{\zeta(\xi)}, \quad e'_M = e_M|_{\zeta(\xi)} + n|_{\zeta(\xi)}\,A_M, \quad \ell' = \ell|_{\zeta(\xi)} - e_M|_{\zeta(\xi)}\,A^M - \frac{1}{2}n|_{\zeta(\xi)}\,A^M A_M. \tag{3.13}$$

3.3. Horizon supertranslations and hypersurface data

We will now study the effect of horizon supertranslations on the hypersurface data of a generic non-expanding horizon. We will describe supertranslations as *active spacetime diffeomorphisms* $F : \mathcal{M} \to \mathcal{M}$ (as opposed to coordinate transformations), which act on the spacetime metric tensor as $g \to F^* g$. That is, F induces a deformation of the metric tensor, while the

[9] The vanishing tensor J_{ab} appearing in the constraint equations (2.12)–(2.14) can so be shown to be left invariant under (3.11) (see [57]).

coordinate charts are left invariant. More specifically, we will characterise horizon supertranslations as diffeomorphisms which leave invariant the horizon as a set of points, and which preserve the full metric tensor on it (see [8, 15, 20, 35–39], [41, 46]). These two conditions can be expressed in a coordinate invariant way as follows

$$\boxed{F(\mathcal{H}) = \mathcal{H}, \qquad \text{and} \qquad F^* g(X,Y)_p = g(X,Y)_p,} \tag{3.14}$$

for all pairs of vectors $X, Y \in T_p\mathcal{M}$ in the spacetime tangent space at points $p \in \mathcal{H}$ on the hypersurface. Actually, we shall see that diffeomorphisms satisfying (3.14) lead to a more general class of transformations than supertranslations, and we will refer to them as *hypersurface symmetries*[10]. We will prove that the action of these diffeomorphisms on the NEH data can be described by the transformations (3.12). That is, hypersurface symmetries, and in particular supertranslations, leave invariant both the intrinsic and the extrinsic geometry of the hypersurface up to a gauge redundancy of the description.

Previous work has used an infinitesimal version of the definition (3.14), which can be recovered expressing F explicitly in (3.14), i.e. in terms of a coordinate system $F : x^\mu \to y^\alpha(x)$ where $\alpha = 0, \ldots, 3$. Then, setting $y^\alpha(x) \approx x^\alpha + \epsilon\, k^\alpha(x)$ and working at linear order in $\epsilon \ll 1$ we find that the second condition in (3.14) reduces to $\mathcal{L}_k g_{\mu\nu}|_{\mathcal{H}} = 0$, which is the definition used in [8, 15, 20, 35–39, 41, 46]. The advantage of (3.14) is that it is coordinate independent, what clarifies the geometrical interpretation of these transformations, and that it can be solved easily leading directly to the finite form of the diffeomorphisms.

It is worth mentioning that hypersurface symmetries have also been studied before in the context of horizon shells [69–71], where they represent the 'soldering' freedom between two spacetimes across a non-expanding null hypersurface [72, 73].

3.3.1. Spacetime coordinate system. In order to find the set of diffeomorphisms satisfying the conditions (3.14) we will first define a spacetime coordinate system adapted to $\mathcal{H}$ to simplify the derivation. As the hypersurface $\mathcal{H}$ is diffeomorphically identified with the abstract manifold Σ via the embedding map $\Phi : \Sigma \to \mathcal{M}$, we can use the coordinate system $\{\xi^a\}$ on Σ to parametrise points over the hypersurface. Then, we can extend these coordinates away from $\mathcal{H}$ introducing a transverse coordinate r, which we define by the conditions $\ell = \partial_r$ and $r(\mathcal{H}) = 0$, and requiring the coordinates $\{\xi^a\}$ to be constant along the integral lines of ℓ. Strictly speaking this procedure would require one to define how the rigging is extended off the hypersurface, but the following calculation is independent of this extension, and thus we will leave it unspecified. The complete spacetime coordinate system is given by $x^\mu \equiv \{u = \xi^1, r, x^M = \xi^M\}$, and therefore it follows that the embedding map is simply

$$\Phi : \xi^a \longrightarrow x^\mu = \{u = \xi^1, r = 0, x^M = \xi^M\}. \tag{3.15}$$

With these choices the corresponding coordinate basis for the spacetime tangent space $\{\partial_u, \partial_r, \partial_M\}$ coincides with the basis $\mathcal{B} = \{e_1, \ell, e_M\}$ defined in section 2.1

$$n = \partial_u, \qquad \ell = \partial_r, \qquad e_M = \partial_M. \tag{3.16}$$

In order to be consistent with our conventions, which require $\boldsymbol{n}(\ell) = 1$, the normal form to the hypersurface must be given by $\boldsymbol{n} = dr$. In addition, from (2.5) it follows that the metric tensor is of the form

[10] As a consistency check for this approach we have also rederived the full BMS group at null infinity—including BMS supertranslations—using similar techniques. See appendix C.1.

$$g_{\mu\nu}(x)|_{r=0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q_{MN} \end{pmatrix}, \qquad \text{with} \qquad q_{MN} = q_{MN}(x^M), \tag{3.17}$$

at points on the hypersurface $\mathcal{H}$, which is located at $r = 0$. In the following we will use '$\hat{=}$' to write equations which hold on the hypersurface $\mathcal{H}$, that is, at $r = 0$. Note also that the normal vector is given $n = g^{-1}(\boldsymbol{n}, \cdot) = e_1$, in consistency with the setting defined in section 2.1. For later reference, note that the first derivatives of the metric have the form

$$\partial_u g_{\mu\nu} \hat{=} 0, \qquad \partial_L g_{\mu\nu} \hat{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_L q_{MN} \end{pmatrix}, \qquad \frac{1}{2}\partial_r g_{\mu\nu} \hat{=} \begin{pmatrix} -\kappa & \cdot & -\Omega_M \\ \cdot & \cdot & \cdot \\ -\Omega_N & \cdot & \Xi_{MN} \end{pmatrix}, \tag{3.18}$$

where the last equality follows directly from the definition of the tensor Y_{ab} (2.8), and the empty entries are those which cannot be determined from the hypersurface data alone. With this information we are ready to find those diffeomorphisms F satisfying the conditions (3.14), and to characterise their action on the hypersurface data.

3.3.2. Hypersurface symmetries. The first condition on (3.14) requires that the diffeomorphism F maps the hypersurface to itself. Since the null normal $\boldsymbol{n}$ of a hypersurface is unique up to a scale, F can only change the normalisation of $\boldsymbol{n}$, that is $F^*\boldsymbol{n} = \lambda_1 \boldsymbol{n}$, where λ_1 is some function on $\mathcal{H}$. In the coordinate system defined above this condition has the explicit form

$$y^\alpha_\mu \, n_\alpha(y(x)) \hat{=} \lambda_1 \, n_\mu(x) \quad \Longleftrightarrow \quad y^1_\mu \hat{=} \lambda_1 \, \delta^1_\mu \quad \Longrightarrow \quad \lambda_1 \hat{=} y^1_r, \tag{3.19}$$

where we are using the shorthand $y^\alpha_\mu \equiv \partial_\mu y^\alpha$. Moreover, since the metric tensor g should also be preserved by F, the normal vector $n = g^{-1}(\boldsymbol{n}, \cdot)$ can only change its normalisation under the action of the diffeomorphism F, i.e. $dF(n) = \lambda_2 \, n$, where λ_2 is some function over $\mathcal{H}$. The explicit form of this condition is

$$\lambda_2 \, n^\alpha(y(x)) \hat{=} y^\alpha_\mu \, n^\mu(x) \quad \Longleftrightarrow \quad \lambda_2 \, \delta^\alpha_u \hat{=} y^\alpha_u \quad \Longrightarrow \quad \lambda_2 \hat{=} y^0_u. \tag{3.20}$$

The following sequence of identities shows that the transformation of the null normal and the normal vector are related by the condition $\lambda_1 = \lambda_2^{-1}$

$$\begin{aligned} 1 &\hat{=} g(\ell, n) \hat{=} F^* g(\ell, n) \hat{=} g(dF(\ell), dF(n)) \hat{=} \lambda_2 \, g(dF(\ell), n) \hat{=} \\ &\lambda_2 \, \boldsymbol{n}(dF(\ell)) \hat{=} \lambda_2 \, F^*\boldsymbol{n}(\ell) \hat{=} \lambda_1 \lambda_2 \, \boldsymbol{n}(\ell) \hat{=} \lambda_1 \lambda_2, \end{aligned} \tag{3.21}$$

where we have used the transformation properties of the metric tensor, the normal form and the normal vector, and the definition of the pullback map (see e.g. [74]). Summarising, the first condition in (3.14) implies that the diffeomorphism should satisfy the following constraints at points on the hypersurface

$$\partial_u y^1 \hat{=} \partial_M y^1 \hat{=} 0, \qquad \partial_u y^I \hat{=} 0, \quad \text{and} \quad \partial_u y^0 \hat{=} 1/\partial_r y^1. \tag{3.22}$$

The second condition in (3.14) is satisfied if and only if the diffeomorphism preserves all the scalar products between the elements of the basis $\mathcal{B}$. From the results obtained above it is straightforward to check that the requirements

$$F^* g(n, \ell) \hat{=} g(n, \ell), \quad F^* g(n, n) \hat{=} g(n, n), \quad \text{and} \quad F^* g(n, e_M) \hat{=} g(n, e_M) \tag{3.23}$$

are satisfied already without imposing further conditions. In order for F to preserve the remaining scalar products the following equations must hold

$$F^*g(e_M,e_N) \hat{=} g(e_M,e_N): \qquad g'_{\alpha\beta}y^\alpha_M y^\beta_N \hat{=} g_{MN} \iff q'_{IJ}\,Y^I_M Y^J_N \hat{=} q_{MN}, \tag{3.24}$$

$$F^*g(\ell,e_M) \hat{=} g(\ell,e_M): \qquad g'_{\alpha\beta}y^\alpha_r y^\beta_M \hat{=} 0 \iff y^I_r \hat{=} -\frac{1}{f_u}f^M Y^I_M, \tag{3.25}$$

$$F^*g(\ell,\ell) \hat{=} g(\ell,\ell): \qquad g'_{\alpha\beta}y^\alpha_r y^\beta_r \hat{=} 0 \iff y^0_r \hat{=} -\frac{1}{2f_u}f^M f_M, \tag{3.26}$$

where we have defined $f(u,x^M) \equiv y^0(x)|_{r=0}$, $Y^I(x^M) \equiv y^I(x)|_{r=0}$, and $f_u \equiv \partial_u f$. The indices M,N are raised and lowered with the metric q_{MN} and its inverse q^{MN}, and we also used a prime to denote quantities evaluated at $y(x)$, e.g. $g'_{\alpha\beta} = g_{\alpha\beta}(y(x))$.

We can conclude that the action of hypersurface symmetries at points of the hypersurface $\mathcal{H}$ is completely determined by the following functions

$$\boxed{y^0(x) \hat{=} f(u,x^M), \qquad y^I(x) \hat{=} Y^I(x^M),} \tag{3.27}$$

where $f(u,\, x^M)$ has an arbitrary dependence on its variables, while the components $Y^I(x^M)$ are independent on u due to (3.22). Moreover, equation (3.24) implies that Y^I must define an isometry of the $u-$ independent metric q_{MN}. Note that the conditions (3.14) only constrain the form of hypersurface symmetries at points of the hypersurface $\mathcal{H}$, and then, their extension away from $\mathcal{H}$ is arbitrary.

The diffeomorphisms we just found are more general than the supertranslations discussed in [15, 35–39, 41, 46], which correspond to the case when the functions (3.27) are of the form

$$f(u,x^M) = u + A(x^M), \qquad \text{and} \qquad Y^I(x^M) = x^I. \tag{3.28}$$

Actually, supertranslations can be singled out noting that, in addition to (3.14), they also also preserve the normalisation of the null normal, namely $F^*\boldsymbol{n} = \boldsymbol{n}$, which implies $f_u = 1$.

3.3.3. Effect on the horizon geometry. To conclude this section we will discuss the effect that hypersurface symmetries have on the data $\mathscr{D} = \{\gamma_{ab}, \ell_a, \ell^{(2)}, Y_{ab}\}$. Note that, since the diffeomorphisms satisfying (3.14) preserve the scalar products on $\mathcal{H}$, they also leave invariant the metric data of the horizon $\{\gamma_{ab}, \ell_a, \ell^{(2)}\}$, given by (2.4) and (2.5), and in particular its intrinsic geometry. Therefore all that remains to compute is the effect of these diffeomorphisms on the geometric data encoded in the tensor Y_{ab}. The form of the tensor Y_{ab} after a horizon supertranslation can be obtained doing the substitution $g \to F^*g$ in its definition (2.8)

$$\begin{aligned} Y'_{ab} &= \frac{1}{2}e^\mu_a e^\nu_b \mathcal{L}_\ell (F^*g)_{\mu\nu} = \frac{1}{2}e^\mu_a e^\nu_b \partial_r \left(g_{\alpha\beta}(y)\; y^\alpha_\mu y^\beta_\nu\right) \\ &= \frac{1}{2}\left(g_{\alpha\beta,\gamma}y^\gamma_r y^\alpha_\mu y^\beta_\nu + g_{\alpha\beta}y^\alpha_{r\mu}y^\beta_\nu + g_{\alpha\beta}y^\alpha_\mu y^\beta_{r\nu}\right) e^\mu_a e^\nu_b, \end{aligned} \tag{3.29}$$

with $e_a = \{n, e_M\}$. The relevant derivatives of the metric tensor have been given in (3.18), the first and second derivatives of $y^\alpha(x)$ can be determined from (3.22) and (3.24)–(3.26), and the basis vector components e^μ_a are defined in (3.16). As in the previous subsection we also assume for simplicity that $Y^I(x^M) = x^I$. After a long but straightforward computation we obtain

$$\begin{aligned} Y'_{nn}(\xi) &= -\kappa|_{\zeta(\xi)}\hat{f}_n - \partial_n \log\hat{f}_n, \\ Y'_{nM}(\xi) &= -\Omega_M|_{\zeta(\xi)} - \kappa|_{\zeta(\xi)}\hat{f}_M - \partial_M \log\hat{f}_n, \\ Y'_{MN}(\xi) &= \frac{1}{\hat{f}_n}(\Xi_{MN}|_{\zeta(\xi)} - \Omega_{(M}|_{\zeta(\xi)}\,\hat{f}_{N)} - \kappa|_{\zeta(\xi)}\hat{f}_M\hat{f}_N - D_M\hat{f}_N) \end{aligned} \tag{3.30}$$

where $\zeta^i(\xi) = (\hat{f}(\xi), \xi^I)$ is a diffeomorphism of the abstract manifold Σ defined as $\zeta \equiv \Phi^{-1} \circ F \circ \Phi$, in terms of the embedding map Φ of the hypersurface (3.15). Note that the diffeomorphism $\zeta(\xi)$ is well defined, as F maps the hypersurface $\mathcal{H}$ onto itself. It is straightforward to identify the behaviour of the tensor Y_{ab} under a hypersurface symmetry with the effect of a residual gauge transformation (3.11). In particular, the action of supertranslations (3.28) on the horizon data is identical to (3.12). Thus, in the following we will make no distinction between horizon supertranslations and the gauge transformations acting as (3.12).

The same conclusion can be reached analysing the effect of hypersurface symmetries in the tensor Y_{ab} in terms of the diffeomorphism ζ, without making use of the explicit spacetime coordinate system constructed above. Using the well-known property $\mathcal{L}_\ell(F^* g) = F^*(\mathcal{L}_{dF(\ell)} g)$ (see e.g. [75]) one finds the following equalities

$$Y' = \frac{1}{2}\Phi^* \mathcal{L}_\ell(F^* g) = \frac{1}{2}(F \circ \Phi)^* \mathcal{L}_{\mathrm{d}F(\ell)} g = \frac{1}{2}(\Phi \circ \zeta)^* \mathcal{L}_{\mathrm{d}F(\ell)} g = \zeta^*(\frac{1}{2}\Phi^* \mathcal{L}_{\mathrm{d}F(\ell)} g). \tag{3.31}$$

The quantity in parentheses in the last term can be regarded as the tensor $Y|_{\zeta(\xi)}$ changed under a rigging transformation (3.3), which is followed by a diffeomorphism in the abstract manifold. These are the two gauge redundancies considered in section 3.2.

The results presented in this section have two main consequences: on the one hand hypersurface symmetries, and in particular supertranslations, leave invariant both the intrinsic and extrinsic geometries of the horizon up to a gauge redundancy. On the other hand, since the residual gauge redundancies (3.11) leave the equations (2.12)–(2.14) unchanged, diffeomorphisms acting as hypersurface symmetries also preserve the form of the constraint equations of null hypersurfaces. That is, supertranslations are a symmetry of the NEH constraint equations.

The next step is to determine the action of these transformations on the dynamical degrees of freedom on the horizon. In other words, we need to find a free data set necessary and sufficient to describe the full horizon geometry, and then we have to characterise the action of supertranslations on such data set. Depending on whether these diffeomorphisms have a nontrivial action on the NEH free data or not, we will identify them as large gauge transformations (i.e. global symmetries of the constraint equations) or pure gauge redundancies of our description. In section 4 we will consider the constraint equations (2.12)–(2.14) for a NEH in order to single out an appropriate free data set, and characterise the geometric nature of horizon supertranslations. For completeness in section 5 we will perform a similar analysis for BMS supertranslations acting on null infinity.

4. Evolution of non-expanding horizons

In the present section we will study the solutions to the constraint equations of non-expanding horizons embedded in vacuum, (2.18) and (2.17). Our main objective is to extract a free data set, $\mathscr{D}_{\text{free}}$, necessary and sufficient to reconstruct the full NEH geometry, and then to determine the behaviour of the free data set under supertranslations. Our starting point is the data set (2.11) presented in section 2.1, which contains *sufficient* information to characterise completely the NEH geometry. However the data set (2.11) still involves some residual gauge freedom, which we characterised in section 3.2. Moreover, the data elements of (2.11) are not freely specifiable, as they are subject to the constraints (2.18) and (2.17).

Our strategy will be, first, to reduce the residual gauge freedom (3.11) imposing appropriate gauge fixing conditions, so that the only remaining ambiguity is (3.12), which we associated to supertranslations in section 3.3. Then, we will turn to the resolution of the constraints (2.18) and (2.17), and we will present a free data set, $\mathscr{D}_{\rm free}$, composed of quantities invariant under supertranslations. Note that, since this free data set involves no unfixed gauge redundancies, all of its elements are *necessary* to describe the NEH geometry. Finally, we will check explicitly that our free data set encodes all the information about the spacetime curvature which was contained in the original data set (2.11). With this result we conclude that the supertranslation invariant data set $\mathscr{D}_{\rm free}$ is both *necessary and sufficient* to reconstruct the NEH geometry, and that supertranslations act trivially on the dynamical variables of the NEH.

In order to simplify the analysis of the gauge redundancies and constraint equations (2.18) let us introduce a more convenient set of variables than (2.11). The hypersurface data element Ω_M defines a one-form on the spatial sections of the horizon $\mathcal{S}_{\xi^1} \cong \mathbb{S}^2$, and therefore we can decompose it uniquely as the sum of an exact part Ω^e_M and a divergence free part Ω^0_M

$$\Omega_M = \Omega^e_M + \Omega^0_M, \qquad \Omega^e_M \equiv \partial_M \eta, \qquad D^M \Omega^0_M = 0 \tag{4.1}$$

where $\eta(\xi)$ is a smooth function of the coordinates. The previous equation represents the Hodge decomposition of Ω_M on $\mathcal{S}_{\xi^1}$, which determines Ω^0_M uniquely and the potential $\eta(\xi)$ is defined up to a shift, $\eta \to \eta + \eta_0$, where $\partial_M \eta_0 = 0$. Due to the properties of the Hodge decomposition, the exact and divergence free part of each side of the Damour–Navier–Stokes equations (2.17) are separately equal, leading to

$$\partial_n \Omega^0_M = 0, \qquad\qquad \partial_n \partial_M \eta = \partial_M \kappa. \tag{4.2}$$

In particular, this implies that the divergence free part of the Hajicek one form is constant along the null direction. Moreover, the equation on the right can be solved in terms of $\eta(\xi)$ requiring it to satisfy $\kappa = \partial_n \eta$, what determines η uniquely up to an additive constant on the horizon. The potential η can also be defined in a more covariant way in terms of a decomposition of the rotation one-form $\omega_a = (\kappa, \Omega_M)$, which we defined in (2.9). More specifically, if ω_a is a solution of (2.17) it can be decomposed as (see [49, 76])

$$\boxed{\omega_a = \partial_a \eta + \omega^0_a, \quad \text{where} \quad \omega^0(\hat{n}) = 0 \quad \text{and} \quad D^M \partial_M \eta = D^M \Omega_M.} \tag{4.3}$$

Here ω^0_a is uniquely determined to be $\omega^0_a = (0, \Omega^0_M)$, and $\eta(\xi)$ is defined up to a constant shift. As we shall see in section 4.2 the potential η will play an essential role to solve (2.18) in terms of quantities which are invariant under supertranslations. Finally, let us also introduce the following combination[11]

$$\boxed{\Sigma^0_{MN} \equiv \frac{1}{2} D_{(M}\Omega_{N)} + \Omega_M \Omega_N + \kappa\, \Xi_{MN},} \tag{4.4}$$

together with its trace $\theta^0 \equiv \Sigma^{0M}_{M}$ and its traceless part σ^0_{MN}. From the definitions (4.3) and (4.4), it is straightforward to check that the NEH data (2.11) can be equivalently encoded in a new set of quantities $\mathscr{D}_s$

$$\mathscr{D} = (q_{MN}, \quad \kappa, \quad \Omega_M, \quad \Xi_{MN}) \quad \longrightarrow \quad \mathscr{D}_s = (q_{MN}, \quad \eta, \quad \Omega^0_M, \quad \Sigma^0_{MN}). \tag{4.5}$$

As we will see below the new data set $\mathscr{D}_s$ has particularly simple transformation properties under supertranslations.

[11] The quantity Σ^0_{MN} can be defined covariantly in terms of the rotation one form ω_a. Using the connection $\overline{\nabla}$ defined by equation (17) of [57] we have $\Sigma^0_{ab} \equiv \frac{1}{2}\overline{\nabla}_{(a}\omega_{b)} + \omega_a \omega_b$.

4.1. Reduction of the gauge freedom

We now introduce the relevant gauge fixing conditions so that the residual gauge freedom (3.11) reduces to the transformations (3.12), which we identified with supertranslations. Given an arbitrary non-expanding horizon with surface gravity κ, it is always possible to choose a gauge where the surface gravity is a constant κ_0 over the horizon

$$\textit{Gauge condition 1}: \qquad \partial_n \kappa_0 = \partial_M \kappa_0 = 0 \quad \text{and} \quad \kappa_0 > 0 \quad \text{for all } \xi \in \Sigma. \tag{4.6}$$

This gauge can be achieved making a transformation of the form (3.11) with the function $\hat{f}(\xi)$ satisfying

$$\kappa(\xi)\hat{f}_n + \partial_n \log \hat{f}_n = \kappa_0, \tag{4.7}$$

which can always be solved for $\hat{f}(\xi)$. It is straightforward to check that, in this gauge, the equation (2.17) implies that the full Hajicek one-form Ω_M—and not only Ω^0_M—must be constant along the null direction of the horizon, $\partial_n \Omega_M = 0$.

The previous gauge fixing condition still does not reduce the redundancies down to supertranslations. Indeed, after imposing the condition (4.6) on the data, the remaining gauge freedom can be found solving again equation (4.7), but this time setting $\kappa(\xi) = \kappa_0$, which gives

$$\hat{f}(\xi) = \xi^1 + A(\xi^M) + \frac{1}{\kappa_0} \log\left(1 + B(\xi^M) \mathrm{e}^{-\kappa_0 \xi_1}\right), \tag{4.8}$$

where $B(\xi^M)$ and $A(\xi^M)$ are smooth functions satisfying $\partial_n A = \partial_n B = 0$. At this point we can already identify $A(\xi^M)$ with the freedom to perform a supertranslation (3.12). Therefore it only remains to find a convention to eliminate ambiguity associated to $B(\xi^M)$, which reflects the fact that the gauge condition (4.7) does not determine completely the normalisation of the null normal $\boldsymbol{n}$. The analysis in the following sections is independent of the actual method to fix the normalisation of $\boldsymbol{n}$, what was discussed for example in [50, 58, 76]. Here we will follow the strategy of [50], which consists in imposing a gauge fixing condition on $\theta^0 = \Xi^M_M$. Contracting the constraint equation for Ξ_{MN} (2.18) with q^{MN}, and using that the surface gravity κ_0 and Ω_M are constant along ξ^1 we find

$$\partial_n \theta^0 = -\kappa_0 \left(\theta^0 - \frac{1}{2}\mathcal{R}\right), \tag{4.9}$$

where the quantity[12] θ^0 was defined in (4.4), and $\mathcal{R}$ is the scalar curvature associated to q_{MN}. Note that in this equation only θ^0 has a non trivial dependence on ξ^1, since $\partial_n q_{MN} = 0$ also implies $\partial_n \mathcal{R} = 0$, and thus it can be integrated easily

$$\theta^0(\xi) = \left(\theta^0|_{\xi^1_0} - \frac{1}{2}\mathcal{R}\right) \mathrm{e}^{-\kappa_0(\xi^1 - \xi^1_0)} + \frac{1}{2}\mathcal{R}. \tag{4.10}$$

As was discussed in [50], for a subclass of non-expanding horizons, known as *generic non-expanding horizons*, it is possible to perform a transformation of the form (4.8) in order to make θ^0 stationary on the horizon

$$\textit{Gauge condition 2}: \qquad \partial_n \theta^0 = 0 \qquad \text{for all } \xi \in \Sigma. \tag{4.11}$$

This condition is trivially satisfied by all black holes in the Kerr family, which are stationary, and thus they have generic horizons (see e.g. appendix D in [58]). As we show in appendix

[12] The quantity θ^0 should not be confused with the trace of the second fundamental form, the expansion $\theta \equiv \Theta^M_M$, which is zero for non-expanding horizons.

B.1, if the hypersurface data of a NEH satisfies the following condition on a spatial section $\mathcal{S}_{\xi^1}$ of Σ

$$\Omega^{0M}\Omega^0_M \leqslant \frac{1}{2}\mathcal{R} \leqslant \theta^0 \qquad \text{for all } \xi \in \mathcal{S}_{\xi^1}, \tag{4.12}$$

the horizon can be shown to be generic. In other words, it is possible to find a gauge transformation of the form (4.8) which allows one to set $\partial_n\theta^0 = 0$ everywhere on Σ. Moreover, the residual gauge freedom left after imposing this gauge fixing condition is precisely that of supertranslations (3.12). To the best of our knowledge the condition (4.12) has not been presented before in the literature. In the following we will restrict ourselves to horizons where the gauge (4.11) can be attained.

After imposing the gauge fixing conditions (4.6) and (4.11), the only remaining gauge freedom are supertranslations (3.12). We will now characterise the behaviour under supertranslations of the elements in the data set $\mathscr{D}_s$ (4.5). The transformation properties of the spatial metric q_{MN} were discussed in section 3.2, and it was shown to be invariant under (3.12). Since the surface gravity κ_0 has been set to a constant, the exact and divergence free parts of the Hajicek one-form transform under supertranslations as

$$\Omega^e_M{}'(\xi) = \Omega^e_M|_{\zeta(\xi)} + \kappa_0 A_M, \qquad \Omega^0_M{}'(\xi) = \Omega^0_M|_{\zeta(\xi)}, \tag{4.13}$$

where $\zeta^a = (\xi^1 + A(\xi^M), \xi^M)$. Actually, due to (4.2) the divergence free part of the Hajicek one-form satisfies $\partial_n\Omega^0_M = 0$, and thus Ω^0_M is completely invariant. In this gauge the functional form of the potential $\eta(\xi)$ reduces to $\eta(\xi) = \kappa_0\xi^1 + h(\xi^M)$, with $\partial_n h = 0$, and it behaves under supertranslations as[13]

$$\eta(\xi) \to \eta'(\xi) = \kappa_0\xi^1 + h(\xi^M) + \kappa_0 A(\xi^M). \tag{4.14}$$

The previous expression can also be written as $\eta'(\xi) = \eta(\zeta(\xi))$, which means that η transforms under a supertranslation as a scalar field. Finally, it is easy to check that the object Σ^0_{MN}, defined in (4.4), also transforms as a scalar under (3.12)

$$\Sigma^0_{MN}{}'(\xi) = \Sigma^0_{MN}|_{\zeta(\xi)}. \tag{4.15}$$

Thus, from (4.13)–(4.15), it follows that all the elements of the NEH data set $\mathscr{D}_s$ (4.5) are either invariant or transform as a scalar field under supertranslations.

4.2. Resolution of the hypersurface constraint equations

In this subsection we will consider the constraint equations of the NEH, and we will present a free data set $\mathscr{D}_{\text{free}}$ composed of quantities which are all invariant under supertranslations.

The data set $\mathscr{D}_s$ (4.5) is subject to the following complete set of constraint equations and gauge fixing conditions

$$\partial_n q_{MN} = 0, \qquad \partial_n\Omega^0_M = 0, \qquad \theta^0 = \frac{1}{2}\mathcal{R}, \qquad \partial_n\sigma^0_{MN} = -\kappa_0\,\sigma^0_{MN}, \tag{4.16}$$

and the potential has to be of the form $\eta(\xi) = \kappa_0\xi^1 + h(\xi^M)$, where $\partial_a\kappa_0 = \partial_n h = 0$, due to (4.6). Let us recapitulate the origin of these equations from left to right: the first one is a consequence of the non-expanding condition and the Raychaudhuri equation, (2.16); the second one follows from the DNS equation (4.2); the third and fourth ones are obtained expressing the equation (2.18) for the transverse connection in terms of Σ^0_{MN} (4.4), and then decomposing it

[13] The behaviour of $\eta(\xi)$ under (3.12) should be derived from its definition in (4.3).

in its trace θ^0, and traceless σ^0_{MN} parts. To obtain the last two equations we also used the gauge fixing conditions (4.6) and (4.11).

The equations (4.16) imply that the full NEH geometry can be reconstructed from an initial data set specified on a spatial section $\mathcal{S}_{\xi^1_0}$ of the horizon

$$(q_{MN}|_{\mathcal{S}_{\xi^1_0}}, \quad \Omega^0_M|_{\mathcal{S}_{\xi^1_0}}, \quad \sigma^0_{MN}|_{\mathcal{S}_{\xi^1_0}}), \tag{4.17}$$

and providing, in addition, the scalar potential $\eta(\xi) = \kappa_0 \xi^1 + h(\xi^M)$. This initial data set, and the quantities κ_0 and $h(\xi^M)$, can be chosen freely on a given spatial slice $\mathcal{S}_{\xi^1_0}$, and then the geometry over the entire NEH can be obtained solving (4.16).

As we discussed above, all the elements in the data set $\mathscr{D}_s$ transform as scalar fields under supertranslations (3.12), implying that the initial data (4.17) may still involve gauge dependent quantities. Let us examine the transformation properties of the elements in (4.17) under supertranslations:

- The gauge freedom (3.12) is defined in terms of *active diffeomorphisms* on Σ, which transform the NEH data but leave the coordinate system unchanged. In consequence, the initial slice $\mathcal{S}_{\xi^1_0}$ (defined by $\xi^1 = \xi^1_0$) does not transform under supertranslations.
- The objects q_{MN} and Ω^0_M do not depend on the null coordinate due to (4.16), and thus $q_{MN}|_{\mathcal{S}_{\xi^1_0}}$ and $\Omega^0_M|_{\mathcal{S}_{\xi^1_0}}$ are invariant under supertranslations.
- The potential $\eta(\xi) = \kappa_0 \xi^1 + h(\xi^M)$ has a non trivial dependence on the null coordinate, and so does σ^0_{MN} unless it is strictly zero (see equation (4.16)). Therefore, in general, both the potential $\eta(\xi)$, and the initial value $\sigma^0_{MN}|_{\mathcal{S}_{\xi^1_0}}$ will transform non-trivially under supertranslations.

At this point, we could impose appropriate gauge fixing conditions to eliminate the ambiguity associated with the transformations[14] (3.12). However, following the strategy used for null infinity in [53], we will deal with this redundancy introducing a supertranslation invariant free data set, and proving that it contains the same information about the spacetime geometry as the original data (2.11). This is a rigorous way to ensure that we do not exclude physically allowed configurations of the NEH.

4.2.1. Supertranslation invariant data. In order to define a free data set which is composed of quantities invariant under supertranslations it is convenient to parametrise the null direction of the horizon using the scalar potential $\eta(\xi)$. Note that this parametrisation is well defined, since $\eta(\xi)$ increases monotonically along the null direction everywhere in Σ, i.e. $\partial_n \eta = \kappa_0 > 0$. Then, using the fact that both $\sigma^0_{MN}(\xi)$ and $\eta(\xi)$ transform as scalar fields under supertranslations, we can construct a supertranslation invariant variable expressing the evolution of σ^0_{MN} along the null direction in terms of η. For this purpose, let us write the null coordinate in terms of η as $\xi^1 = H(\eta, \xi^M)$,

$$\eta(\xi) = \kappa_0 \xi^1 + h(\xi^M) \qquad \Longrightarrow \qquad H(\eta, \xi^M) = \frac{1}{\kappa_0}\big(\eta - h(\xi^M)\big). \tag{4.18}$$

As the potential $\eta(\xi)$ changes under supertranslations (4.14), the inverse function $H(\eta, \xi^M)$ needs to transform accordingly

[14] This is the approach used in the membrane paradigm for the description of black holes (e.g. see appendix D in [62]). Other examples of this method are reviewed in [58].

$$H(\eta,\xi^M) \to H'(\eta,\xi^M) = \frac{1}{\kappa_0}\big(\eta - h(\xi^M)\big) - A(\xi^M). \tag{4.19}$$

Thus, we can characterise the evolution of σ^0_{MN} along the null coordinate in terms of the following *supertranslation invariant* variable

$$\boxed{s_{MN}(\eta,\xi^M) \equiv \sigma^0_{MN}\big(H(\eta,\xi^M),\xi^M\big).} \tag{4.20}$$

To prove that this object is invariant under (3.12) we just need to use its definition in combination with the transformation properties of σ^0_{MN} and the function $H(\eta,\xi^M)$ under (3.12)

$$\begin{aligned} s'_{MN}(\eta,\xi^M) &= \sigma^0_{MN}{}'\big(H'(\eta,\xi^M),\xi^M\big) = \sigma^0_{MN}\big(H'(\eta,\xi^M) + A(\xi^M),\xi^M\big) \\ &= \sigma^0_{MN}\big(H(\eta,\xi^M),\xi^M\big) = s_{MN}(\eta,\xi^M). \end{aligned} \tag{4.21}$$

Therefore, the transformed form of $s'_{MN}(\eta,\xi^M)$ after a supertranslation is the same function of η as $s_{MN}(\eta,\xi^M)$, which proves that this object is completely invariant under the action of supertranslations.

Following a similar line of argument it can also be shown that s_{MN} is invariant under gauge transformations (3.11) with $\zeta^a(\xi) = (\lambda\xi^1,\xi^M)$, where λ is a constant over the horizon.

4.2.2. Solution to the constraint equations. The constraint equation for $s_{MN}(\eta,\xi^M)$ is obtained expressing the last equation in (4.16) in terms of η and s_{MN}. Using that $\partial_\eta H = 1/\kappa_0$ we find

$$\partial_\eta s_{MN} = \partial_\eta H\, \partial_n \sigma^0_{MN}|_{H(\eta)} \qquad \Longrightarrow \qquad \partial_\eta s_{MN} = -s_{MN}, \tag{4.22}$$

which has the general solution

$$s_{MN}(\eta,\xi^M) = s_{MN}|_{\eta_0}\, \mathrm{e}^{-\,(\eta-\eta_0)}, \tag{4.23}$$

and η_0 is an arbitrary constant which reflects the ambiguity in the definition of η. This ambiguity can also be eliminated imposing an additional condition on the data, e.g. the normalisation

$$\frac{1}{a_{\mathcal{H}}}\oint_{\mathcal{S}_{\eta=0}} \mathrm{d}\xi^2 q\; s_{MN}s^{MN} = 1, \tag{4.24}$$

where the integral is over the spatial section $\mathcal{S}_{\eta=0}$ of the horizon, $q \equiv \sqrt{\det(q_{MN})}$ and $a_{\mathcal{H}}$ is the area of $\mathcal{S}_{\eta=0}$.

The result (4.23), together with the equations (4.16), imply that the full NEH geometry can be encoded in the functional form of the potential $\eta = \kappa_0\xi^1 + h(\xi^M)$, combined with the following free data set

$$\boxed{\text{Free horizon data :} \qquad \mathscr{D}_{\text{free}} \equiv \big(q_{MN}|_{\mathcal{S}_{\eta_0}},\quad \Omega^0_M|_{\mathcal{S}_{\eta_0}},\quad s_{MN}|_{\mathcal{S}_{\eta_0}}\big),} \tag{4.25}$$

which is specified on a spatial slice of the horizon $\mathcal{S}_{\eta_0}$ defined by $\eta = \eta_0$. The full NEH geometry can be recovered from these quantities using that q_{MN} and Ω^0_M are constant along the null direction of the horizon and (4.23). In particular, $q_{MN}|_{\mathcal{S}_{\eta_0}}$ determines the intrinsic geometry of the NEH, and $\Omega^0_M|_{\mathcal{S}_{\eta_0}}$ can be associated to its angular momentum aspect when q_{MN} admits an $SO(2)$ isometry (see [58]). It is also interesting to note that s_{MN} (which is symmetric and traceless) has two independent components, matching the number of radiative degrees of freedom of the gravitational field.

Due to (4.16), the first two elements of (4.25), $q_{MN}|_{\mathcal{S}_{\eta_0}} = q_{MN}|_{\mathcal{S}_{\xi^1_0}}$ and $\Omega^0_M|_{\mathcal{S}_{\eta_0}} = \Omega^0_M|_{\mathcal{S}_{\xi^1_0}}$, coincide with the first two elements in (4.17), which we argued to be invariant under supertranslations.

Moreover, the third element $s_{MN}|_{\mathcal{S}_{\eta_0}}$ is also invariant under (3.12) due to (4.21), implying that none of the elements in (4.25) involve any unfixed gauge freedom. As a consequence, distinct data sets $\mathscr{D}_{\text{free}}$ generate gauge inequivalent NEH structures, and thus, the corresponding spacetime geometries must be different as well. In other words, all the elements in $\mathscr{D}_{\text{free}}$ are *necessary* to characterise completely the geometry of the NEH. Note, however, that the potential $\eta = \kappa_0 \xi^1 + h(\xi^M)$ is also part of the NEH initial data set, and it transforms non-trivially under supertranslations (4.14).

4.3. Free horizon data and the spacetime curvature

We will now show explicitly that the free data set $\mathscr{D}_{\text{free}}$ (4.25) encodes all the information about the curvature of the ambient spacetime $\mathcal{M}$ contained in the original data set (2.11). In other words, the supertranslation invariant data set (4.25) is both *necessary and sufficient* to characterise the entire NEH geometry, and thus no knowledge about the functional form of the potential $\eta(\xi)$ is required. In addition, using the Newman–Penrose formalism, we will argue that $\mathscr{D}_{\text{free}}$ is sufficiently general to describe radiative processes taking place at the horizon.

In the case of horizons embedded in vacuum, $T_{\mu\nu} = 0$, the curvature of the ambient spacetime is completely described by the Weyl tensor $R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}$, or equivalently, by the five Weyl scalars Ψ_n, with $n = 0, \ldots, 4$. Since the connection coefficients (2.7) only characterise the spacetime connection along the horizon they only constrain four of the Weyl scalars. Indeed, the computation of Ψ_4 requires knowledge about the spacetime connection off the hypersurface, which is not available in (2.7). Therefore, all we need to show is that all the information about the spacetime curvature contained in the Weyl scalars Ψ_n, with $n = 0, \ldots, 3$ is also encoded in the free data set $\mathscr{D}_{\text{free}}$ (4.25).

The computation of the Weyl scalars can be done as described in section 2.3. First, without loss of generality, we specify an arbitrary point ξ_0^M on the spatial sections $\mathcal{S}_{\xi^1}$ of the horizon, choosing the coordinates ξ^M such that[15] $q_{MN}(\xi_0) = \delta_{MN}$. Then, the Weyl scalars can be obtained from (2.21) contracting the Weyl tensor with the elements of the Newman–Penrose tetrad $\mathcal{B}_{NP} = \{n, \ell, m, \overline{m}\}$. The explicit expressions for the Weyl scalars $\Psi_n(\xi^1, \xi^M)$ are functions of the coordinates ξ^a, and therefore they are not invariant under diffeomorphisms of the abstract manifold. However, if we impose the gauge fixing conditions (2.5), (4.6) and (4.11) the only remaining gauge transformations are supertranslations, $\xi^1 \to \xi^1 + A(\xi^M)$. Thus, in order to deal with this freedom we introduce the *gauge corrected* Weyl scalars

$$\Psi_n^c(\eta, \xi^M) \equiv \Psi_n(H(\eta, \xi^M), \xi^M). \tag{4.26}$$

In the case of non-expanding horizons embedded in vacuum these quantities read

$$\boxed{\Psi_0^c = \Psi_1^c = 0, \qquad \Psi_2^c = -\frac{1}{4}\mathcal{R} + \frac{\mathrm{i}}{2}\mathcal{J}, \qquad \text{with} \qquad \mathcal{J} \equiv D_{[2}\Omega^0_{3]},} \tag{4.27}$$

and

$$\Psi_3^c = \frac{1}{\kappa_0\sqrt{2}}\Big[Ds|_{\eta_0}\, \mathrm{e}^{-(\eta-\eta_0)} + \hat{D}\Psi_2^c + 3\,\hat{\Omega}^0\Psi_2^c + 3\,\hat{\Omega}^e\Psi_2^c\Big]. \tag{4.28}$$

Here we have used the notation $\hat{D} \equiv (D_2 - \mathrm{i}D_3)$, and $\hat{\Omega} \equiv (\Omega_2 - \mathrm{i}\Omega_3)$. We have also defined the complex field

[15] This choice determines our gauge fixing conventions (2.5) at $\xi^a = \xi_0^a$. As discussed in section 3 this choice is preserved by the residual gauge redundancies (3.11), including supertranslations.

$$Ds(\eta, \xi^M) \equiv D^M s_{M2} + \Omega^{0|M} s_{M2} - \mathrm{i}(D^M s_{M3} + \Omega^{0|M} s_{M3}). \tag{4.29}$$

The details of the computation can be found in appendix B.2. The Weyl scalars Ψ_0^c and Ψ_1^c, represent gravitational radiative modes which propagate into the horizon, and their vanishing can be seen as a consistency condition for the horizon to be non-expanding. The scalar Ψ_2^c encodes the coulomb contribution of the gravitational field and, when q_{MN} admits an axial killing vector field, $\mathcal{J}$ characterises the angular momentum aspect of the NEH (see [58]). Finally, the Weyl scalars Ψ_3^c and Ψ_4^c can be associated to radiative modes propagating along the horizon.

As we explained above, the structure of the NEH does not constrain the value of the fourth Weyl scalar, and thus *a priori* Ψ_4^c can take any value on $\mathcal{H}$. Since we also have $\Psi_0^c = \Psi_1^c = 0$ and $\Psi_2^c, \Psi_3^c \neq 0$, we can conclude that the Weyl tensor on a NEH will be generically of Petrov type II (see [58, 66]). Then, in general, the gravitational field on the NEH will contain a radiative component [51]. In other words, the NEH structure is sufficiently general to allow for the presence of gravitational radiation on the horizon.

It is straightforward to check that Ψ_0^c, Ψ_1^c and Ψ_2^c are invariant under supertranslations, and that Ψ_2^c can be computed from the elements in $\mathscr{D}_{\text{free}}$. However, the expression (4.28) for Ψ_3^c still involves gauge dependent quantities which are not part of the free data set (4.25), namely, the surface gravity κ_0, which depends on the normalisation of the null normal, $\boldsymbol{n}$, and the exact part of the Hajicek one form Ω_M^e which transforms under supertranslations. We will now show that both quantities can be associated to well known gauge redundancies of the Newman–Penrose formalism, i.e. the freedom to perform rotations of the null tetrad $\mathcal{B}_{NP}$. That is, neither κ_0 or Ω_M^e involve any information about the spacetime geometry. More specifically, we will prove that the expression (4.28) for Ψ_3^c represents the same spacetime geometry as

$$\boxed{\Psi_3^c = \frac{1}{\sqrt{2}}\Big[Ds|_{\eta_0}\,\mathrm{e}^{-(\eta-\eta_0)} + \hat{D}\Psi_2^c + 3\,\hat{\Omega}^0\,\Psi_2^c\Big],} \tag{4.30}$$

which is completely determined by the elements of $\mathscr{D}_{\text{free}}$. Both expressions for the third Weyl scalar (4.28) and (4.30) are projections of the *same Weyl tensor* associated to two different null tetrads $\mathcal{B}_{NP}$.

We will begin considering the surface gravity κ_0. The Newman–Penrose formalism has an inherent gauge freedom associated to the choice of null tetrad $\mathcal{B}_{NP}$ which, in a general setting, is only required to satisfy the orthogonality and normalisation conditions (2.20). Thus, when there are no further restrictions, it is possible to perform the following redefinition of the null tetrad $\mathcal{B}_{NP}$ which preserves (2.20)

$$n' = \lambda n, \qquad \ell' = \lambda^{-1}\ell, \qquad m' = m, \qquad \overline{m}' = \overline{m}, \tag{4.31}$$

where λ is real scalar field on Σ. From (2.21) it is immediate to check that if we transform the null tetrad as in (4.31)—keeping the Weyl tensor fixed—the gauge corrected Weyl scalars behave as (see section 8 in [66])

$$\Psi_0^{c\prime} = \Psi_0^c = 0, \qquad \Psi_1^{c\prime} = \Psi_1^c = 0, \qquad \Psi_2^{c\prime} = \Psi_2^c, \qquad \Psi_3^{c\prime} = \lambda^{-1}\Psi_3^c. \tag{4.32}$$

That is, Ψ_n^c and $\Psi_n^{c\prime}$ represent contractions of the same Weyl tensor with the elements of two different tetrads, $\{n, \ell, m, \bar{m}\}$ and $\{n', \ell', m', \bar{m}'\}$, respectively, and thus the two sets of Weyl scalars describe the same spacetime geometry. In our setting, the null tetrad is fully determined by the elements in the basis $\mathcal{B}$ adapted to $\mathcal{H}$, and thus the rotations (4.31) must always be associated to a gauge transformation (3.11) for consistency with the definition of $\mathcal{B}$. Actually, it is possible to implement a rotation of the null tetrad of the form (4.31) performing a transformation (3.11) with $\zeta^a(\xi) = (\lambda\xi^1, \xi^M)$, where $\lambda > 0$ is an arbitrary positive constant. The corresponding change in the hypersurface data can be derived from (3.11), and the definition of s_{MN} (4.20)

$$\mathcal{R}' = \mathcal{R}, \qquad \kappa_0' = \lambda\,\kappa_0, \qquad Ds' = Ds, \qquad \hat{\Omega}' = \hat{\Omega}. \tag{4.33}$$

Then, using this data to compute the transformed Weyl scalars (4.27) and (4.28) it is straightforward to check that the behaviour of Ψ_n^c under these transformations is precisely (4.32), i.e. it is indistinguishable from the effect of a null tetrad rotation. This proves explicitly that gauge transformations with $\zeta^a(\xi) = (\lambda\xi^1, \xi^M)$ and $\partial_a\lambda = 0$ leave invariant the spacetime geometry, and thus κ_0 can be set to any arbitrary value in (4.28) without changing the geometric information encoded in Ψ_3^c.

We will now discuss the role of the exact part of the Hajicek one-form Ω_M^e in (4.28). Consider the following redefinition of the null tetrad $\mathcal{B}_{NP}$ which preserves the scalar products (2.20)

$$n' = n, \qquad m' = m + a\,n, \qquad \overline{m}' = \overline{m} + \bar{a}\,n, \qquad \ell' = \ell - \bar{a}\,m - a\,\overline{m} - a\bar{a}\,n, \tag{4.34}$$

where $a = a_2(\xi) + \mathrm{i}a_3(\xi)$ is a complex valued function on Σ. Under this change of null tetrad, and keeping the Weyl tensor fixed, Ψ_n^c behave as (see [66])

$$\Psi_0^{c\prime} = \Psi_0^c = 0, \qquad \Psi_1^{c\prime} = \Psi_1^c = 0, \qquad \Psi_2^{c\prime} = \Psi_2^c, \qquad \Psi_3^{c\prime} = \Psi_3^c + 3\bar{a}\,\Psi_2^c. \tag{4.35}$$

In our framework it can be shown, using (2.19) and (3.13), that supertranslations $\xi^1 \to \xi^1 + A(\xi^M)$ induce a rotation of the null tetrad $\mathcal{B}_{NP}$ of the form (4.34) with $a \equiv (A_2 + \mathrm{i}A_3)/\sqrt{2}$. Moreover, from (3.12) it follows that supertranslations act on the data appearing in (4.27) and (4.28) as

$$\mathcal{R}' = \mathcal{R}, \qquad \kappa_0' = \kappa_0, \qquad Ds' = Ds, \qquad \hat{\Omega}^{e\prime} = \hat{\Omega}^e + \kappa_0\sqrt{2}\,\bar{a}. \tag{4.36}$$

These transformations lead precisely to the behaviour of the Weyl scalars described by (4.35) when we apply them to (4.27) and (4.28). Therefore, the effect of a supertranslation in the Newman–Penrose formalism is entirely equivalent to a rotation of the null tetrad, which has no effect on the horizon geometry. As a consequence, the quantity $\hat{\Omega}^e$ could be changed to any value in (4.28) without affecting the information about the spacetime curvature carried by Ψ_3^c, e.g. it could be eliminated from (4.28) choosing $\bar{a} = -\hat{\Omega}^e/(\kappa_0\sqrt{2})$ in (4.34). This concludes our proof that the two expressions (4.28) and (4.30) for the third Weyl scalar Ψ_3^c can be identified as projections of *the same Weyl tensor* expressed in terms of two different null tetrads. As a consequence (4.28) and (4.30) represent the same spacetime geometry, which can be entirely encoded in the supertranslation invariant free data set $\mathscr{D}_{\text{free}}$.

Summarising, we have argued that the horizon free data set (4.25) is both *necessary and sufficient* to reconstruct all the information about the spacetime geometry determined by the NEH structure:

- the free data set $\mathscr{D}_{\text{free}}$ (4.25) involves no gauge degrees of freedom,
- all the information about the spacetime curvature which is contained in (2.11) is also encoded in $\mathscr{D}_{\text{free}}$, and
- the corresponding data about the curvature tensor can be recovered using the expressions of the Weyl scalars (4.27) and (4.30), and the relation $R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}$ which holds in vacuum.

Since the all the elements in the data set $\mathscr{D}_{\text{free}}$ are invariant under supertranslations, this result completes the proof that horizon supertranslations act trivially on the NEH geometry.

5. Radiative vacua of null infinity

For completeness, in this section we review the role of BMS supertranslations at null infinity using Penrose's conformal framework, and the intrinsic description of $\mathcal{I}$ developed in [52, 53] (see also [25, 26, 54]). In particular, we will reproduce the well-known result that the radiative vacuum of asymptotically flat spacetimes is degenerate, and we will discuss the connection of this degeneracy with supertranslations. At null infinity the would-be gauge degree of freedom associated to BMS supertranslations is *necessary* to have a complete characterisation of the dynamics of $\mathcal{I}$, i.e. it cannot be gauged away. Thus, BMS supertranslations act non-trivially on the geometric data of $\mathcal{I}$.

The difference between the dynamical behaviour of a NEH and null infinity can be traced back to three main causes. First, the structure of null infinity is only defined up to conformal transformations, what requires that the dynamical degrees of freedom of $\mathcal{I}$ are encoded in appropriate *equivalence classes of data sets*. Second, the Ricci tensor of the conformal completion of the physical spacetime does not satisfy the ordinary Einstein's equations, implying that the constraint equations we used for NEH's, (2.17) and (2.18), are no longer valid for null infinity. And finally, contrary to the case of horizons, the boundary conditions for the gravitational field at null infinity allow for gravitational radiation propagating in a transverse direction to reach $\mathcal{I}$.

The geometry of null infinity will be described using the same formalism as in the case of non-expanding horizons, and thus the following analysis shall serve as a non-trivial consistency check of our approach.

5.1. Asymptotically flat spacetimes

Let us begin recalling the definition of asymptotic flatness and null infinity following [53]. A spacetime $(\hat{\mathcal{M}}, \hat{g})$ is said to be asymptotically flat at null infinity if it is possible to find a spacetime $(\mathcal{M}, g)$, together with an embedding $\Psi : \hat{\mathcal{M}} \to \mathcal{M}$, and a function Ω on $\mathcal{M}$ such that

(i) $\Psi^* g_{ab} = \Omega^2\, \hat{g}_{ab}$ on $\hat{\mathcal{M}}$.
(ii) $I \cong \mathbb{S}^2 \times \mathbb{R}$ is the boundary of $\Psi(\hat{\mathcal{M}})$ on $\mathcal{M}$, located at $\Omega = 0$.
(iii) The normal form is given by $n_\mu \equiv \nabla_\mu \Omega \neq 0$ on I.
(iv) There is a neighbourhood of I on $\mathcal{M}$, such that $\hat{g}_{ab}$ satisfies the vacuum Einstein equations, i.e. $\hat{R}_{ab} = 0$.

The spacetime $(\mathcal{M}, g)$ is called the unphysical spacetime, and the hypersurface $I \subseteq \mathcal{M}$, which is null as a consequence of (i), (ii) and (iv), is referred as null infinity. Note that, if the pair (Ω, g) defines an appropriate conformal completion, so does the pair $(\omega\,\Omega, \omega^2\, g)$ for some smooth positive function[16] ω on $\mathcal{M}$. Two asymptotic completions related in this way are regarded as equivalent, and the freedom to perform such conformal transformations should be considered as a gauge redundancy.

5.2. Hypersurface data of null infinity

To describe the geometry of the null hypersurface I we can use the formalism introduced in section 2.1. Thus, we introduce an abstract manifold $\mathcal{I}$, which acts as a diffeomorphic copy of $I \subseteq \mathcal{M}$ detached from the unphysical spacetime, and the identification is performed via the embedding $\Phi : \mathcal{I} \to \mathcal{M}$, such that $\Phi(\mathcal{I}) = I$. We will also choose the coordinate system ξ^a

[16] the factors of ω are chosen so that the physical metric $\Omega^{-2} g$ remains the same.

for $\mathcal{I}$ and the rigging ℓ following the conventions in section 2.1, so that the hypersurface data can be represented by the set of quantities (2.11). In the case of null infinity it is possible to simplify the hypersurface data taking advantage of the freedom to perform conformal transformations $(\Omega, g) \to (\omega\Omega, \omega^2 g)$. Actually, under this change the normal form is rescaled as $\boldsymbol{n} \to \omega\,\boldsymbol{n}$, what can be used to require that the normal vector n^μ satisfies [52] (see also [77])

$$\nabla_\nu n^\mu = 0 \ \text{ on } I \qquad \Longrightarrow \qquad \kappa = \Omega_M = \Theta_{MN} = 0, \tag{5.1}$$

where the conditions on the connection coefficients follow from (2.7). In this gauge, the second fundamental form vanishes, and therefore null infinity $\mathcal{I}$ admits a description as a non-expanding null hypersurface with $\partial_n q_{MN} = 0$. In addition, using the same conformal freedom, we can impose that q_{AB} describes a two dimensional metric of constant scalar curvature $\mathcal{R}$ [52]. In this setting the hypersurface data of $\mathcal{I}$ has the form

$$\gamma_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & q_{MN} \end{pmatrix}, \qquad \ell^a = (1, 0, 0), \qquad \ell^{(2)} = 0, \qquad Y_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \Xi_{MN} \end{pmatrix}. \tag{5.2}$$

In the following, to simplify the notation, we will make no distinction between null-infinity I and its abstract copy $\mathcal{I}$.

5.3. Residual gauge redundancies

The conventions introduced above do not eliminate all the gauge redundancies of our description. Regarding the conformal transformations, the present setting fixes completely the normalisation of the null vector n. However, we are still allowed to perform conformal transformations with a conformal factor that satisfies $\omega|_I = 1$ at null infinity, but takes arbitrary values away from it. From the definition of the hypersurface data it is straightforward to check that under this residual conformal transformations the metric data $\{\gamma_{ab}, \ell_a, \ell^{(2)}\}$ remain invariant, but the transverse components of the tensor Y_{ab} transform as

$$\Xi'_{MN} = \Xi_{MN} + \lambda\, q_{MN}, \tag{5.3}$$

where $\lambda(\xi) \equiv \mathcal{L}_\ell \omega|_{\mathcal{I}}$ can be any smooth function on $\mathcal{I}$.

In addition to these conformal transformations, our description of null infinity also involves redundancies associated to the freedom to perform diffeomorphisms on the abstract manifold, and the choice of rigging. Actually, the analysis of the gauge redundancies of the hypersurface data that we presented in section 3 is also applicable here, since the condition (5.1) implies that null infinity can be described a non-expanding null hypersurface, and the conventions (5.2) are the same we used to study horizons. Taking the results of section 3 into account, and recalling that the normalisation of the null normal $n = \partial_{\xi^1}$ is fixed by our choice of conformal gauge, we find that the only residual freedom of this type are BMS supertranslations[17] (3.12)

$$\Xi'_{MN} = \Xi_{MN} - D_M A_N, \tag{5.4}$$

which corresponds to a diffeomorphism of the abstract manifold $\mathcal{I}$ acting as $\zeta^a(\xi) = (\xi^1 + A(\xi^M), \xi^M)$. Note that, since $\kappa = 0$, these transformations leave invariant all other elements of the hypersurface data. We will denote the group of BMS supertranslations by $\mathcal{S}$.

[17] Similarly to the case of horizons, BMS supertranslations can be described as hypersurface symmetries of null infinity $I \subseteq \mathcal{M}$ (see appendix C.1).

It might seem surprising that we did not encounter the BMS group when discussing the gauge freedom at null infinity. The reason is that we fixed the scale of the null normal (by equation (5.1), and choosing q_{MN} to describe a sphere with curvature $\mathcal{R}$) before characterising the residual gauge freedom of our description. Indeed, the BMS group can be recovered as the set of gauge transformations that leave invariant the conventions (2.5) up to a conformal transformation $(\Omega, g) \to (\omega\Omega, \omega^2 g)$, with $\omega \neq 1$ at $\mathcal{I}$. When the conformal transformations are gauge fixed so that only those with $\omega|_{\mathcal{I}} = 1$ are allowed, the remaining residual gauge is given by the transformations (5.3) and (5.4) that we described above. A derivation of the full BMS group can be found in appendix C.1, where we perform a similar analysis to that of section 3.3 for non-expanding horizons. In the appendix we show that the BMS group can be described as the set of diffeomorphisms of the unphysical spacetime which leaves invariant the metric tensor $g_{\mu\nu}$ and the null normal $\boldsymbol{n}$ of $\mathcal{I}$ up to a conformal transformation, $g \to \omega^2 g$ and $\boldsymbol{n} \to \omega \boldsymbol{n}$.

5.4. Constraint equations

As in the case of non-expanding horizons, the hypersurface data of null infinity cannot be specified freely. It must be consistent with the constraint equations (2.12- 2.14), which are mathematical identities satisfied by any null hypersurface. In the previous paragraphs we have presented most of the elements involved in these equations, and it only remains to compute the terms (2.15). The crucial difference with our previous discussion of non-expanding horizons is that, although the physical spacetime is vacuum in a neighbourhood of $\mathcal{I}$ (condition (*iv*)), the *unphysical Ricci tensor* $R_{\mu\nu}$ *does not vanish*. This is just a direct consequence of the non-trivial transformation properties of the Ricci tensor under the conformal rescaling of the metric (see [77]). Therefore, (2.17) and (2.18) are not valid for $\mathcal{I}$.

In the following subsection we will characterise the terms (2.15) of the constraint equations of null infinity, i.e. the unphysical Ricci tensor $R_{\mu\nu}$, and then we will turn to the resolution of the constraints in section 5.6.

5.5. The Ricci tensor at null infinity

In order to compute the Ricci tensor at points of null infinity it is convenient to note that the Weyl tensor $C_{\rho\sigma\mu\nu}$ is vanishing at $\mathcal{I}$. This implies that the unphysical Riemann tensor at null infinity has the general form [52, 53]

$$R_{\sigma\rho\mu\nu} = \frac{1}{2}(g_{\sigma[\mu}S_{\nu]\rho} - g_{\rho[\mu}S_{\nu]\sigma}), \tag{5.5}$$

where the symmetric tensor $S_{\mu\nu}$ is the Schouten tensor defined at the beginning of section 2. The tensor $S_{\mu\nu}$ has a particularly simple form when expressed in the basis $\mathcal{B} = \{n, \ell, e_M\}$ due to our gauge fixing conventions (5.1) and (5.2). Indeed, the divergence free condition (5.1) can be used in combination with the Ricci identity to prove that the four components $S_{na} \equiv n^\mu e_a^\nu S_{\mu\nu}$ vanish at null infinity

$$\begin{aligned} 0 &= \ell_\sigma n^\mu e_A^\nu \nabla_{[\mu}\nabla_{\nu]} n^\sigma = \ell_\sigma n^\mu e_A^\nu R^\sigma_{\ \rho\mu\nu} n^\rho = \frac{1}{2} S_{An}, \\ 0 &= q^{AB}\, e_{A|\sigma} n^\mu e_B^\nu \nabla_{[\mu}\nabla_{\nu]} n^\sigma = q^{AB}\, e_{A|\sigma} n^\mu e_B^\nu R^\sigma_{\ \rho\mu\nu} n^\rho = -S_{nn}. \end{aligned} \tag{5.6}$$

Moreover, it is also possible to show that the components of the Schouten tensor satisfy $S^M_M = \mathcal{R}$, where $\mathcal{R}$ is the scalar curvature of q_{MN}. This expression can be derived comparing the result of computing $R_{MANB}q^{AB}q^{MN}$ directly from (5.5), with the outcome of the same computation using the identity (B.33) (see appendix B.2) together with the gauge conditions (5.1). We can simplify $S_{\mu\nu}$ even further making use of the residual conformal transformations with $\omega|_{\mathcal{I}} = 1$, which act on the Schouten tensor as (see [52])

$$S'_{ab} = S_{ab}, \qquad S'_{\ell a} = S_{\ell a} - 2\partial_a \lambda, \qquad S'_{\ell\ell} = S_{\ell\ell} - 2\mu + 4\lambda^2, \tag{5.7}$$

where $\lambda(\xi) = \nabla_\ell\, \omega|_{\mathcal{I}}$ and $\mu(\xi) = \ell^\mu \ell^\nu \nabla_\mu \nabla_\nu\, \omega|_{\mathcal{I}}$ are two arbitrary functions on $\mathcal{I}$. Therefore, we can set $S_{\ell\ell} = 0$ by a suitable choice of the function μ.

Collecting these results, and using the inverse metric (2.6) to compute the trace of the Schouten tensor, it is possible to derive the Ricci tensor of the unphysical spacetime using $R_{\mu\nu} = S_{\mu\nu} + \frac{1}{3} S g_{\mu\nu}$. We find the following non-vanishing components

$$\boxed{R_{n\ell} = S_{n\ell}, \quad R_{MN} = S_{MN} + \frac{1}{2}(2S_{n\ell} + \mathcal{R}) q_{MN},} \tag{5.8}$$

and $R_{nn} = R_{nM} = R_{\ell\ell} = 0$. This form for the unphysical Ricci tensor at null infinity is universal for any asymptotically flat spacetime. We will now derive the additional conditions satisfied by $R_{\mu\nu}$ in regions of $\mathcal{I}$ where no outgoing radiation is present.

5.5.1. Geometry of the radiative vacuum. In order to find the relevant boundary conditions we need to consider the leading order contribution $K_{\rho\sigma\mu\nu} \equiv \Omega^{-1} C_{\rho\sigma\mu\nu}$ to the Weyl tensor, since the unphysical Weyl tensor $C_{\rho\sigma\mu\nu}$ always vanishes on $\mathcal{I}$ [52].

The condition that there is no outgoing radiation in a region of $\mathcal{I}$ is most easily expressed in terms of the leading order Weyl scalars, which are defined as components of the tensor $K_{\sigma\rho\mu\nu}$ in the basis $\mathcal{B}_{NP} = \{\ell, n, m, \overline{m}\}$ (see [65]). Note that we have changed the order of the first two elements of the null tetrad $\mathcal{B}_{NP}$, ℓ and n, with respect to section 4.3, while m and $\overline{m}$ are defined by (2.19). The relevant Weyl scalars are given by

$$\Psi_2^0 = K^\mu_{\ \nu\rho\sigma} \ell_\mu m^\nu n^\rho \overline{m}^\sigma, \qquad \Psi_3^0 = K^\mu_{\ \nu\rho\sigma} \ell_\mu n^\nu n^\rho \overline{m}^\sigma, \qquad \Psi_4^0 = K^\mu_{\ \nu\rho\sigma} \overline{m}_\mu n^\nu \overline{m}^\rho n^\sigma, \tag{5.9}$$

and the conditions for no outgoing radiation at a region of $\mathcal{I}$ read [26]

$$\textit{Radiative vacuum}: \qquad \mathrm{Im}\Psi_2^0 = 0, \qquad \Psi_3^0 = 0, \quad \text{and} \quad \Psi_4^0 = 0. \tag{5.10}$$

The implications of these boundary conditions on the form of the Schouten tensor can be derived from the equations

$$\textit{Bianchi Identities}: \qquad \nabla_{[\mu} S_{\nu]\sigma} = -K_{\mu\nu\sigma\rho} n^\rho, \tag{5.11}$$

which are a direct consequence of Bianchi identities of the unphysical spacetime [26, 52]. In order to solve the previous equations and boundary conditions, it is convenient to express them in the basis $\mathcal{B} = \{\ell, n, e_M\}$. Taking contractions on both sides of (5.11) with appropriate combinations of the elements in $\mathcal{B}$, and using (2.7) in combination with the gauge conditions (5.1) to simplify the result, we find

$$\mathrm{Im}\Psi_2^0 = 0 \qquad \Longrightarrow \qquad D_{[M} S_{N]\ell} = \Xi^P_{[M} S_{N]P}, \tag{5.12}$$

$$\Psi_3^0 = 0 \qquad \Longrightarrow \qquad \partial_{[n} S_{M]\ell} = 0, \quad \text{and} \quad D_{[M} S_{N]P} = 0, \tag{5.13}$$

$$\Psi_4^0 = 0 \qquad \Longrightarrow \qquad \partial_n S_{MN} = 0. \tag{5.14}$$

A detailed derivation can be found in appendix C.2. The last equation (5.14) implies that the components S_{MN} have to be constant along the null direction of $\mathcal{I}$. Moreover, the form of S_{MN}

can be found solving the second constraint in (5.13) in combination with $S_M^M = \mathcal{R}$, and it has the unique solution

$$S_{MN} = \frac{1}{2}\mathcal{R}\, q_{MN}. \tag{5.15}$$

The original proof can be found in [52], but given that the setting therein is slightly different from ours, for completeness we have written a summary of it in appendix C.2. From the previous relation it follows that the rhs of (5.12) must vanish, which together with the first equation in (5.13), also implies that the components $S_{a\ell}$ take the form $S_{a\ell} = \partial_a S_\ell$ for some function S_ℓ on $\mathcal{I}$. Thus, from equation (5.7) it is straightforward to check that in the radiative vacuum the components $S_{a\ell}$ are pure conformal gauge.

With these results at hand, we can finally obtain the components of the unphysical Ricci tensor on $\mathcal{I}$ *in the absence of outgoing radiation*

$$\boxed{R_{n\ell} = \partial_n S_\ell, \qquad R_{MN} = (\partial_n S_\ell + \mathcal{R})q_{MN},} \tag{5.16}$$

and $R_{nn} = R_{nM} = R_{\ell\ell} = 0$. This result will allow us to write down the constraint equations at regions of $\mathcal{I}$ where there is no outgoing radiation, and whose solutions represent the radiative vacua of asymptotically flat spacetimes.

5.6. Constraint equations for the transverse connection

Before discussing the constraint equations let us comment on the physical degrees of freedom contained in the transverse connection Ξ_{MN}. At the beginning of this section we identified the trace of Ξ_{MN} as a pure conformal gauge (see equation (5.3)), and thus, in order to eliminate this redundancy we will proceed as in [53], identifying those connections related by a conformal transformation. In other words, we will introduce the equivalence relation

$$\Xi_{MN} \approx \Xi'_{MN} \quad \Longleftrightarrow \quad \Xi'_{MN} - \Xi_{MN} = \lambda\, q_{MN}, \tag{5.17}$$

where $\lambda(\xi)$ is an arbitrary smooth function on $\mathcal{I}$, and we will work with the resulting equivalence classes. This amounts to neglecting the trace part of the transverse connection Ξ_{MN}, leaving as the dynamical field its traceless part $\Xi_{MN} - \frac{1}{2}\Xi_L{}^L q_{MN}$. Note that this quantity describes precisely two degrees of freedom, which can be identified with the two radiative degrees of freedom of gravitational radiation [26, 53].

5.6.1. General form of the constraint equations.

We begin discussing the constraint equations for a general situation in the presence of radiation. In particular, we will show that the two components in the traceless part of Ξ_{MN} are both necessary and sufficient to describe the radiative degrees of freedom of the gravitational field at null infinity.

In the presence of radiation at null infinity, the terms (2.15) in the constraint equations can be computed from the Ricci tensor given in (5.8)

$$J_{nn} = J_{nM} = 0, \qquad J_{MN} = -S_{MN} - S_{n\ell}q_{MN} - \frac{1}{2}\mathcal{R}q_{MN}. \tag{5.18}$$

This result together with the gauge conditions (5.1) imply that the Raychaudhuri (2.12) and Damour–Navier–Stokes equations (2.13) are trivially satisfied on $\mathcal{I}$. The only non-trivial equations are those for the transverse components of the connection (2.14), which can be expressed as

$$\boxed{\partial_n \Xi_{MN} - \frac{1}{2}(S_{MN} + S_{n\ell} q_{MN}) \approx -\frac{1}{2} N_{MN},} \tag{5.19}$$

where we have used the equivalence relation (5.17). Here $N_{MN} \equiv S_{MN} - \frac{1}{2}\mathcal{R} q_{MN}$, is the *news* tensor, which vanishes in the absence of radiation passing through $\mathcal{I}$, i.e. when equation (5.15) hold. In addition to the previous equation, the connection must satisfy one more constraint coming from the identity (B.33)

$$D_{[M}\Xi_{N]P} = \frac{1}{2} q_{P[M} S_{N]\ell}. \tag{5.20}$$

Using the Bianchi identities satisfied by the Schouten tensor (5.11) it can be checked that this condition is consistent with the time evolution given by (5.19) (see appendix C.2). In other words, if the previous equation is satisfied at any given value of the null coordinate, then equation (5.19) ensures that it will hold for all values of ξ^1.

We can now show that the two components in the traceless part of Ξ_{MN} encode the radiative modes of gravitational radiation at null infinity. Recall that the information about the outgoing radiative modes at $\mathcal{I}$ is described by the Weyl scalars $\mathrm{Im}\Psi_2^0$, Ψ_3^0 and Ψ_4^0 [26, 53]. As we review in appendix C.2, using the Bianchi identity (5.11), it is possible to express these scalars as

$$\mathrm{Im}\Psi_2^0 = -\frac{1}{2}(D_{[3}S_{2]\ell} - \Xi^M_{[3}S_{2]M}), \tag{5.21}$$

$$\Psi_3^0 = \frac{1}{\sqrt{2}}(D_{[3}S_{2]3} - \mathrm{i}D_{[2}S_{3]2}), \tag{5.22}$$

$$\Psi_4^0 = \frac{1}{2}(\partial_n S_{22} - \partial_n S_{33}) - \mathrm{i}\partial_n S_{23}, \tag{5.23}$$

implying that they are completely determined by the components of the Schouten tensor $S_{M\ell}$ and S_{MN}, and by the transverse connection Ξ_{MN}. Actually, it is easy to prove that only the traceless part of Ξ_{MN} contributes in the first equation, and that these expressions are invariant under the conformal transformations (5.3) and (5.7). Then, the constraint equations (5.19) and (5.20) can be solved for S_{MN} and $S_{M\ell}$ giving

$$S^M_M = \frac{1}{2}\mathcal{R}, \qquad S_{MN} \approx -2\partial_n \Xi_{MN}, \qquad S_{N\ell} = q^{PM} D_{[M}\Xi_{N]P}. \tag{5.24}$$

In these equations too only the traceless part of Ξ_{MN} contains relevant information about the geometry at $\mathcal{I}$, as the contribution of the trace Ξ^M_M can be identified as pure conformal gauge. Thus, we can conclude that the traceless part of Ξ_{MN} is both *necessary and sufficient* to recover completely the information about the radiative modes at $\mathcal{I}$ (for a more detailed derivation see [53]).

5.6.2. Degeneracy of the radiative vacuum. Finally, we turn to the discussion of the degeneracy of the radiative vacua in asymptotically flat spacetimes. The constraint equations for regions of $\mathcal{I}$ with no outgoing radiation can obtained from (5.19) and (5.20) together with the boundary conditions (5.16), which imply $S_{a\ell} = \partial_a S_\ell$, and $N_{MN} = 0$. Using the equivalence relation (5.17) they read

$$\partial_n \Xi_{MN} \approx 0, \qquad D_{[M}\Xi_{N]P} \approx 0. \tag{5.25}$$

Note that Ξ_{MN} still transforms under supertranslations. In order to characterise the set of radiative vacua avoiding possible gauge artifacts we need to introduce a new gauge invariant dynamical variable. Due to our choice of conformal gauge the Hajicek one-form and the surface gravity are vanishing, and therefore we cannot proceed as in the case of NEHs and construct a supertranslation invariant variable analogous to $s_{MN}(\eta)$ in equation (4.20). Instead, following [53], we choose a reference vacuum $\mathring{\Xi}_{MN}$ and then we consider the differences $\Sigma_{MN} = \Xi_{MN} - \mathring{\Xi}_{MN}$, between a generic vacuum connection Ξ_{MN} and the fiducial connection $\mathring{\Xi}_{MN}$. It is easy to check that Σ_{MN} is invariant under supertranslations. Then, given a fixed fiducial connection $\mathring{\Xi}_{MN}$, the set of distinct Σ_{MN} consistent with the equations (5.25) is isomorphic to the set of radiative vacua. Since both of the connections Ξ_{MN} and $\mathring{\Xi}_{MN}$ describe a radiative vacuum, their difference Σ_{MN} also satisfies (5.25) due to the linearity of the equations. The general solution to (5.25), and therefore, the set of radiative vacua of null infinity is characterised by the expression

$$\boxed{\Sigma_{MN} \approx D_M f_N - \frac{1}{2}\Delta f q_{MN},} \tag{5.26}$$

where $f(\xi^M)$ is any smooth function of the coordinates ξ^M (see appendix C.2). It is important to stress that the smoothness $f(\xi^M)$ is essential for the derivation, which uses the fact that the spatial sections of $\mathcal{I}$ are compact and simply connected. We will denote the set of vacuum connections by $\mathring{\Gamma}$.

The previous expression (5.26) already indicates clearly that the set of vacuum connections is infinitely degenerate. Comparing (5.26) with (5.4) it is straightforward to check that, given a fiducial vacuum $\mathring{\Xi}_{MN}$, the most general vacuum connection is given by

$$\Xi_{MN} \approx \mathring{\Xi}_{MN} + D_M f_N - \frac{1}{2}\Delta f q_{MN}, \tag{5.27}$$

and therefore, the difference between any two vacuum connections has the form of a supertranslation. In other words, we can construct the full set $\mathring{\Gamma}$ acting on $\mathring{\Xi}_{MN}$ with all the elements of the group of BMS supertranslations $\mathcal{S}$ (5.4). Note that *BMS translations*, i.e. the four dimensional subgroup $\mathcal{T} \subseteq \mathcal{S}$ of supertranslations satisfying

$$D_M f_N - \frac{1}{2}\Delta f q_{MN} = 0, \tag{5.28}$$

acts trivially on the connections of null infinity. Thus, the set of radiative vacua is isomorphic to the group of supertranslations modulo BMS translations $\mathring{\Gamma} \cong \mathcal{S}/\mathcal{T}$.

It is interesting to see how the presence of a non-vanishing news tensor induces a change of the radiative vacuum. Consider a solution of (5.19) where the news N_{MN} is non-zero in the interval $\xi^1 \in (\xi^1_i, \xi^1_f)$ and vanishes everywhere else. Then, the initial and final states of the connection, $\Xi_{MN}|_{\xi^1_i}$ and $\Xi_{MN}|_{\xi^1_f}$ respectively, represent radiative vacua of $\mathcal{I}$, and have to be of the form (5.27). Integrating (5.19) we find that the difference between the final and initial transverse connections Ξ_{MN} is given by the expression

$$\delta\Sigma_{MN} \equiv \Sigma_{MN}|_{\xi^1_f} - \Sigma_{MN}|_{\xi^1_i} = \Xi_{MN}|_{\xi^1_f} - \Xi_{MN}|_{\xi^1_i} \approx -\frac{1}{2}\int_{\xi^1_i}^{\xi^1_f} \mathrm{d}\xi^1 N_{MN}, \tag{5.29}$$

which is invariant under supertranslations. For a generic source of radiation the configuration of the news tensor N_{MN} will be such that $\delta\Sigma_{MN} \neq 0$ and therefore, in general, the initial and final transverse connections correspond to *distinct radiative vacua*. In particular, it is now clear that if we imposed a gauge fixing condition on Ξ_{MN} to eliminate the freedom to perform supertranslations (5.4) we would be restricting the allowed dynamics at null infinity.

From the discussion in the previous paragraphs we can see that, in contrast to the case of horizons, null infinity supertranslations transform the dynamical variables of $\mathcal{I}$. On the one hand, supertranslations have been shown to act non-trivially on the traceless part of the transverse connection Ξ_{MN} (see equation (5.4)). On the other hand, the two components in the traceless part of Ξ_{MN} are both *necessary and sufficient* to describe the two degrees of freedom of gravitational radiation at $\mathcal{I}$. Thus, connections related by a supertranslation cannot be identified with each other, as this would require gauging away one further component of the traceless part of Ξ_{MN}. As a consequence, BMS supertranslations must be regarded as large gauge transformations, i.e. as *global symmetries* of the constraint equations of null infinity, which act non-trivially on the geometric data of $\mathcal{I}$.

6. Results and discussion

One of the most interesting features about asymptotically flat spacetimes is the infinite dimensional asymptotic symmetry group at null infinity, the BMS group. The BMS symmetries, and in particular null infinity supertranslations, were originally characterised as diffeomorphisms which preserved certain coordinates conventions in a neighbourhood of null infinity [21–23]. Many years later, the study of the geometrical structure of null infinity led to the isolation of the radiative degrees of freedom of the gravitational field [53], and it was understood that BMS supertranslations act non-trivially on the radiative degrees of freedom. Actually, the radiative vacuum of asymptotically flat spacetime was shown to be infinitely degenerate, and that it was possible to transform each of these vacua into any other with a supertranslation.

Recently, it has been argued that the ASG of spacetimes containing a non-extremal black hole should be enhanced with horizon supertranslations. These diffeomorphisms would transform the state of the black hole horizon in an analogous way as BMS supertranslations act on the geometric data of null infinity. According to this proposal, the multiplicity of black hole states generated by horizon supertranslations could provide a partial explanation for the Bekenstein–Hawking entropy formula.

The task of characterising the ASG of the near horizon geometry for non-extremal black holes has been addressed in many works. However, there is no consensus regarding the structure of the ASG, or the physical interpretation of these diffeomorphisms. In the present paper we have presented a detailed characterisation of the geometric properties of supertranslations defined on a generic non-expanding horizon embedded in vacuum. For this purpose we have used a coordinate independent approach analogous to the one used in [52, 53] to study the structure of null infinity in exact, non-linear, general relativity. In this framework, the intrinsic and extrinsic geometry of the horizon are encoded in tensor fields living on an abstract three-dimensional manifold Σ, which acts as a diffeomorphic copy of the horizon separated from the physical spacetime. In particular, the corresponding set of tensor fields, known as *the horizon data set*, contains the dynamical degrees of freedom of the horizon, i.e. the freely specifiable and gauge invariant data of the horizon. In order to extract the dynamical degrees of freedom from the data set, and determine their behaviour under supertranslations, we have followed the strategy described below:

1. First, we have characterised in detail *all the gauge redundancies* in our description of the NEH.

In particular, we have shown that the action of supertranslations on the horizon data is identical to that of a gauge redundancy: they are associated with a reparametrisation of the null direction of the horizon, and a change of the transversal direction used to define the extrinsic geometry, (i.e. the rigging). Thus, supertranslations leave invariant both the intrinsic and extrinsic geometry of the horizon up to a gauge redundancy of the description.

2. To determine the free data of the horizon we have solved the constraints imposed by the vacuum Einstein's equations on the NEH geometry.

As a result of this analysis we have identified the set of geometric quantities which can be freely specified on the horizon, and which encode all the information about the spacetime curvature contained on the NEH geometry. This *free data set* encodes the dynamical degrees of freedom of the horizon, but typically still involves some gauge redundancies.

3. The previous two analyses can be combined to characterise the gauge redundancies on the free data set. This allows one to extract a set of quantities which are both *necessary and sufficient* to reconstruct the full NEH geometry.

This procedure has led us to find a free data set which contains no unfixed gauge degrees of freedom, and in particular, which only involves objects *invariant under supertranslations*. More specifically, this free data set is composed of quantities defined on a particular spatial section $\mathcal{S}_{\eta_0}$ of the horizon

$$\boxed{\text{Free horizon data :} \qquad \mathscr{D}_{\text{free}} \equiv (q_{MN}|_{\eta_0}, \quad \Omega^0_M|_{\eta_0}, \quad s_{MN}|_{\eta_0}).}$$

where q_{MN} represents the induced metric on the spatial sections of the horizon, and Ω^0_M determines its angular momentum aspect when q_{MN} has an $SO(2)$ isometry. In those situations when there is gravitational radiation propagating along the horizon the symmetric traceless tensor s_{MN} can be associated to radiative degrees of freedom of the gravitational field. Since the elements of the free data set $\mathscr{D}_{\text{free}}$ are all invariant under supertranslations, we conclude that *supertranslations act trivially on the NEH geometry*, i.e. they must be regarded as pure gauge. In particular, the stationary state of the NEH, which corresponds to the case $s_{MN}|_{\eta_0} = 0$, is uniquely determined by q_{MN} and Ω^0_M, and it does not transform under supertranslations.

A fundamental step to obtain the supertranslation-invariant data set $\mathscr{D}_{\text{free}}$ is the choice of an appropriate parametrisation for the null direction of the horizon. Rather than using an arbitrary coordinate, the null direction is parametrised by the value of a potential η, which is defined in a coordinate invariant way (see equation (4.3)). The horizon can be foliated by the level sets of the potential η, the spatial sections $\mathcal{S}_\eta$, and the evolution of the geometry along the null direction can be expressed in terms of the dependence on η of the horizon data. In particular, q_{MN} and Ω^0_M are both constant along the null direction of the horizon, while s_{MN} behaves as

$$s_{MN} = s_{MN}|_{\eta_0}\, \mathrm{e}^{-\,(\eta-\eta_0)},$$

which shows that any deviation away from the stationary state, i.e. $s_{MN} = 0$, relaxes exponentially fast to it. The use of the potential to express the evolution of the horizon data avoids the ambiguity associated to supertranslations, which are related to coordinate reparametrisations of the null direction.

It is important to remark that the present work is restricted to the case the non-expanding horizons embedded in vacuum, and thus we have not considered processes involving matter or radiation falling across the horizon. To check if our results can be extended to more general situations we have considered the possibility of 'implanting' supertranslation hair on an event horizon with a non-spherical shock-wave of null matter or radiation, as proposed in [34]. We have found that, consistently with the conclusions of this paper, the shock-wave cannot excite the degree of freedom associated to supertranslations. In other words, the supertranslation degree of freedom cannot encode any 'memory' about the energy–momentum tensor of the shock-wave, which is in harmony with our identification of supertranslations as a gauge redundancy. The corresponding analysis will be presented in a companion paper [59].

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Appendix A. Constraint equations for null hypersurfaces

In this appendix we will present a derivation of the constraint equations for null hypersurfaces (2.12)–(2.14).

A.1. Raychaudhuri equation

We begin discussing the constraint equation (2.12) for the expansion θ of the hypersurface. From the definition of the second fundamental form $\Theta_{MN} = e^{\mu}_{M} e^{\nu}_{N} \nabla_{\mu} n_{\nu}$, and using Leibniz rule to expand the derivative we find

$$\partial_n \Theta_{MN} = e^{\nu}_{N} \nabla_n e^{\mu}_{M} \nabla_{\mu} n_{\nu} + e^{\mu}_{M} \nabla_n e^{\nu}_{N} \nabla_{\mu} n_{\nu} + e^{\mu}_{M} e^{\nu}_{N} n^{\sigma} \nabla_{\sigma} \nabla_{\mu} n_{\nu}, \tag{A.1}$$

where $\nabla_n = n^{\mu} \nabla_{\mu}$. Substituting the connection coefficients (2.7) we have

$$\begin{aligned} \partial_n \Theta_{MN} &= \Theta^{P}_{(M} \Theta_{N)P} + e^{\mu}_{M} e^{\nu}_{N} n^{\sigma} \nabla_{\sigma} \nabla_{\mu} n_{\nu} \\ &= \Theta^{P}_{(M} \Theta_{N)P} - R_{nMnN} + e^{\mu}_{M} e^{\nu}_{N} n^{\sigma} \nabla_{\mu} \nabla_{\sigma} n_{\nu}, \end{aligned} \tag{A.2}$$

where the second equality follows from using the Ricci identity. Using again the Leibniz rule and (2.7) the last term can be written as

$$\begin{aligned} e^{\mu}_{M} e^{\nu}_{N} n^{\sigma} \nabla_{\mu} \nabla_{\sigma} n_{\nu} &= \partial_M (e^{\mu}_{N} \nabla_n n_{\mu}) - \nabla_M e^{\nu}_{N} \nabla_n n_{\nu} - e^{\nu}_{N} \nabla_M n^{\sigma} \nabla_{\sigma} n_{\nu} \\ &= \kappa \Theta_{MN} - \Theta^{P}_{M} \Theta_{NP}. \end{aligned} \tag{A.3}$$

The contribution from the Riemann tensor can be expressed in terms of the Weyl and Ricci tensors

$$R_{nMnN} = C_{nMnN} + \frac{1}{2} S_{nn} q_{MN} = C_{nMnN} + \frac{1}{2} R_{nn} q_{MN}, \tag{A.4}$$

where we are using the shorthand $R_{nMnN} = R_{\mu\nu\rho\sigma} n^{\mu} e^{\nu}_{M} n^{\rho} e^{\sigma}_{N}$, and similar expressions to denote the contraction of spacetime tensors with the elements of the basis $\mathcal{B} = \{n, \ell, e_M\}$. Using the last equation we obtain

$$\partial_n \Theta_{MN} = \kappa \Theta_{MN} + \Theta^{P}_{M} \Theta_{NP} - C_{nMnN} - \frac{1}{2} R_{nn} q_{MN}, \tag{A.5}$$

which is known as the *tidal force equation* (see [58]). The equation for the expansion $\theta = \Theta^M_M$ can be calculated from the expression

$$\partial_n\theta = \partial_n(q^{MN}\Theta_{MN}) = \partial_n q^{MN}\Theta_{MN} + q^{MN}\partial_n\Theta_{MN}. \tag{A.6}$$

Note also that $\partial_n q^{MN} = -2\Theta^{MN}$, what follows from differentiating $q_{MN}q^{NP} = \delta^P_M$ with respect to ξ^1 and using the relation $\Theta_{MN} = \frac{1}{2}\partial_n q_{MN}$. Collecting all these results we arrive to the final expression for the Raychaudhuri equation

$$\partial_n\theta - \kappa\theta + \Theta^{MN}\Theta_{MN} = -R_{nn}. \tag{A.7}$$

A.2. Damour–Navier–Stokes equations

In order to derive the Damour–Navier–Stokes equation (2.13) we will compute the component R_{nA} of the Ricci tensor in terms of the hypersurface data. Using the expression (2.6) for the inverse metric we find

$$R_{nA} = R_{\mu n\nu A}g^{\mu\nu} = R_{\ell nnA} + q^{MN}R_{MnNA}. \tag{A.8}$$

The two terms can be rewritten using the Ricci identity

$$R_{\ell nnA} = \ell^\sigma n^\mu e^\nu_A\nabla_\mu\nabla_\nu n_\sigma - \ell^\sigma n^\mu e^\nu_A\nabla_\nu\nabla_\mu n_\sigma, \tag{A.9}$$

$$R_{MnNA} = e^\sigma_M e^\mu_N e^\nu_A\nabla_\mu\nabla_\nu n_\sigma - e^\sigma_M e^\mu_N e^\nu_A\nabla_\nu\nabla_\mu n_\sigma. \tag{A.10}$$

Each of these four terms can be expressed in terms of the connection coefficients using the definitions (2.7) and the Leibniz rule for the covariant derivative. For example, noting that $\Omega_M = \ell^\sigma e^\nu_M\nabla_\nu n_\sigma$ we have

$$\begin{aligned}\ell^\sigma n^\mu e^\nu_M\nabla_\mu\nabla_\nu n_\sigma &= \partial_n\Omega_M - \nabla_n e^\nu_M\ell^\sigma\nabla_\nu n_\sigma - e^\nu_M\nabla_n\ell^\sigma\nabla_\nu n_\sigma\\ &= \partial_n\Omega_M - \kappa\Omega_M - \Omega^N\Theta_{MN} + \kappa\Omega_M + \Omega^N\Theta_{MN}.\end{aligned} \tag{A.11}$$

Also since $\kappa = \ell^\sigma n^\nu\nabla_\nu n_\sigma$ we have

$$\begin{aligned}\ell^\sigma n^\mu e^\nu_M\nabla_\nu\nabla_\mu n_\sigma &= \partial_M\kappa - \nabla_M n^\mu\ell^\sigma\nabla_\mu n_\sigma - n^\mu\nabla_M\ell^\sigma\nabla_\mu n_\sigma\\ &= \partial_M\kappa - \Omega_M\kappa - \Theta^N_M\Omega_M + \kappa\Omega_M.\end{aligned} \tag{A.12}$$

Collecting terms we find

$$R_{\ell nnM} = R_{nM\ell n} = \partial_n\Omega_M - \partial_M\kappa + \Theta^N_M\Omega_N. \tag{A.13}$$

Similarly it can be shown that

$$R_{MnNA} = D_{[N}\Theta_{A]M} + \Theta_{M[N}\Omega_{A]}. \tag{A.14}$$

Then we have

$$R_{nA} = \partial_n\Omega_A - \partial_A\kappa + D_N\Theta^N_A - D_A\theta + \theta\Omega_A, \tag{A.15}$$

which can be identified with the Damour–Navier–Stokes equations (2.13).

A.3. Equation for the transverse connection

We now describe the derivation of the constraint equation (2.14) for the transverse connection Ξ_{MN}. From the definition of the transverse connection $\Xi_{MN} = \frac{1}{2}e^\mu_{(M}e^\nu_{N)}\nabla_\mu\ell_\nu$, and using Leibniz rule to expand the derivative we find

$$2\partial_n \Xi_{MN} = \nabla_n e^{\mu}_{(M}\, e^{\nu}_{N)} \nabla_\mu \ell_\nu + e^{\mu}_{(M} \nabla_n e^{\nu}_{N)} \nabla_\mu \ell_\nu + e^{\mu}_{(M} e^{\nu}_{N)} \nabla_n \nabla_\mu \ell_\nu. \tag{A.16}$$

Substituting the expressions for the connection coefficients (2.7) we arrive to

$$\begin{aligned}\partial_n \Xi_{MN} &= -2\Omega_M \Omega_N + \Theta^P_{(M} \Xi_{N)P} + \frac{1}{2} e^{\mu}_{(M} e^{\nu}_{N)} \nabla_n \nabla_\mu \ell_\nu \\ &= -2\Omega_M \Omega_N + \Theta^P_{(M} \Xi_{N)P} + \frac{1}{2} R_{(N\ell nM)} + \frac{1}{2} e^{\nu}_{(N} n^\sigma \nabla_{M)} \nabla_\sigma \ell_\nu,\end{aligned} \tag{A.17}$$

where we have also used the Ricci identity to derive the second equality. Using the Leibniz rule for the connection and (2.7) we can rewrite the last term as

$$\begin{aligned}e^{\nu}_{(N} n^\sigma \nabla_{M)} \nabla_\sigma \ell_\nu &= \nabla_{(M} (n^\sigma e^{\nu}_{N)} \nabla_\sigma \ell_\nu) - \nabla_n e_\nu \nabla_{(M} e^{\nu}_{N)} - \nabla_{(M} n^\sigma \nabla_\sigma \ell_\nu e^{\nu}_{N)} \\ &= -D_{(M} \Omega_{N)} - 2\kappa \Xi_{MN} + 2\Omega_M \Omega_N - \Theta^P_{(M} \Xi_{N)P}\end{aligned} \tag{A.18}$$

where D_M is the Levi-Civita connection of the spatial metric q_{MN}. Thus, substituting the previous expression in it we have

$$\partial_n \Xi_{MN} = -\frac{1}{2} D_{(M} \Omega_{N)} - \Omega_M \Omega_N - \kappa \Xi_{MN} + \frac{1}{2} \Theta^P_{(M} \Xi_{N)P} + \frac{1}{2} (R_{N\ell nM} + R_{M\ell nN}). \tag{A.19}$$

Using the symmetries of the Riemann tensor, and the form (2.6) for the inverse metric, we find that

$$\frac{1}{2} (R_{N\ell nM} + R_{M\ell nN}) = -\frac{1}{2} R_{MN} + \frac{1}{2} R_{MANB} q^{AB}, \tag{A.20}$$

where $R_{AB} = g^{\mu\nu} R_{\mu A \nu B}$. We will now rewrite R_{MANB} in terms of the connection coefficients. From the Ricci identity we have

$$R_{MANB} = e_{M|\sigma} \nabla_{[\mu} \nabla_{\nu]} e^\sigma_A\, e^\mu_N e^\nu_B = \partial_{[N} (e_{M|\sigma} \nabla_B e^\sigma_{A]}) - \nabla_{[N} e_{M|\sigma} \nabla_B e^\sigma_{A]} - \nabla_{[N} e^\nu_{B]} \nabla_\nu e^\sigma_A e_{M|\sigma}. \tag{A.21}$$

Substituting the expressions for the connection coefficients (2.7) we obtain

$$R_{MANB} = \partial_{[N} (\overline{\Gamma}^L_{A]B} q_{ML}) - \Theta_{M[N} \Xi_{A]B} - \Xi_{M[N} \Theta_{A]B} - \overline{\Gamma}^L_{M[N} \overline{\Gamma}^P_{A]B} q_{PL}, \tag{A.22}$$

which after contracting with q^{AB} can be written as

$$q^{AB} R_{MANB} = \mathcal{R}_{MN} + \Xi_{P(M} \Theta^P_{N)} - \theta \Xi_{MN} - \Theta_{MN} \theta^\ell. \tag{A.23}$$

Here $\mathcal{R}_{MN}$ is the Ricci tensor associated to the spatial metric q_{MN}. This result can be used together with (A.19) to express (A.17) as follows

$$\begin{aligned}\partial_n \Xi_{MN} &= -\frac{1}{2} D_{(M} \Omega_{N)} - \Omega_M \Omega_N - (\kappa + \frac{1}{2}\theta) \Xi_{MN} + \Theta^P_{(M} \Xi_{N)P} \\ &= -\frac{1}{2} \Theta_{MN} \theta^\ell + \frac{1}{2} \mathcal{R}_{MN} - \frac{1}{2} R_{MN},\end{aligned} \tag{A.24}$$

which leads to the constraint equation for the transverse connection (2.14) after using the identity $\mathcal{R}_{MN} = \frac{1}{2} \mathcal{R} q_{MN}$.

Appendix B. Calculations for non-expanding horizons

B.1. Fixing the normalisation of the null normal

In this appendix we will show that when the NEH data (4.5) satisfy one of the following conditions on a spatial slice $\mathcal{S}_{\xi^1}$

$$
\begin{aligned}
&(i) && \Omega^{0M}\Omega^0_M \leqslant \frac{1}{2}\mathcal{R} \leqslant \theta^0, \\
&(ii) && \Omega^{0M}\Omega^0_M \leqslant \theta^0 \leqslant \frac{1}{2}\mathcal{R},
\end{aligned}
\tag{B.1}
$$

it is possible to find a gauge transformation (3.11) that sets $\partial_n\theta^0 = 0$ on the horizon Σ. Moreover, the gauge freedom that remains after imposing this condition is precisely that of supertranslations.

B.1.1. Conditions to set $\partial_n\theta^0 = 0$. Among the residual gauge freedom (3.11), the transformations $\zeta^a(\xi) = (\hat{f}(\xi), \xi^M)$ that keep κ constant (i.e. gauge condition 1 (4.6)) are given by

$$
\hat{f}(\xi) = \xi^1 + A(\xi^M) + \frac{1}{\kappa_0}\log\left(1 + B(\xi^M)\mathrm{e}^{-\kappa_0\xi^1}\right), \tag{B.2}
$$

$$
\hat{f}_n = \frac{1}{1 + B\mathrm{e}^{-\kappa_0\xi^1}}, \quad \hat{f}_M = A_M + \frac{B_M\mathrm{e}^{-\kappa_0\xi^1}}{\kappa_0(1 + B\mathrm{e}^{-\kappa_0\xi^1})}, \quad \hat{f}_{nM} = -\frac{B_M\mathrm{e}^{-\kappa_0\xi^1}}{(1 + B\mathrm{e}^{-\kappa_0\xi^1})^2}.
$$

Under these transformations the data changes as follows

$$
\kappa'(\xi) = \kappa|_{\zeta(\xi)}, \tag{B.3}
$$

$$
\Omega'_M(\xi) = \Omega_M|_{\zeta(\xi)} + \kappa_0 A_M, \tag{B.4}
$$

$$
\begin{aligned}
\Xi'_{MN}(\xi) = {} & (1 + B\mathrm{e}^{-\kappa_0\xi^1})\left(\Xi_{MN}|_{\zeta(\xi)} - \Omega_{(M}|_{\zeta(\xi)}\, A_{N)} - \kappa_0 A_M A_N - D_M A_N\right) \\
& - \frac{\mathrm{e}^{-\kappa_0\xi^1}}{\kappa_0}\left(\Omega_{(M}|_{\zeta(\xi)}\, B_{N)} + \kappa_0 A_{(M}B_{N)} + D_M B_N\right).
\end{aligned}
\tag{B.5}
$$

The transformation of the quantity Σ^0_{MN}, which is invariant under (3.12), can be found by plugging the previous equations into its definition (4.4), giving

$$
\begin{aligned}
{\Sigma^0_{MN}}' = {} & \Sigma^0_{MN} + B\mathrm{e}^{-\kappa_0\xi^1}\left(\Sigma^0_{MN} - \frac{1}{2}D_{(M}\Omega'_{N)} - \Omega'_M\Omega'_N\right) \\
& - \mathrm{e}^{-\kappa_0\xi^1}\left(\Omega'_{(M}B_{N)} + D_M B_N\right).
\end{aligned}
\tag{B.6}
$$

Taking the trace we find

$$
\theta^{0\prime} = \theta^0 + B\mathrm{e}^{-\kappa_0\xi^1}\left(\theta^0 - \Omega'^M\Omega'_M - D^M\Omega'_M\right) - \mathrm{e}^{-\kappa_0\xi^1}\left(2\Omega'^M B_M + D^M B_M\right). \tag{B.7}
$$

This quantity evolves according to (4.9), and thus, if at any given time (e.g. $\xi^1 = 0$) we can set $\theta^{0\prime} = \frac{1}{2}\mathcal{R}$, then $\partial_{n'}\theta^{0\prime} = 0$ for all ξ^1, where n' is the vector resulting from the transformation of n under the map ζ. This requires to solve

$$D^M B_M + 2\Omega'^M B_M + \left(D^M \Omega'_M + \Omega'^M \Omega'_M - \theta^0\right) B - (\theta^0 - \frac{1}{2}\mathcal{R}) = 0. \tag{B.8}$$

To simplify this expression we decompose the Hajicek one-form Ω_M in its exact $\Omega^e_M = \partial_M \eta$ and divergence free parts $\Omega^0_M = \varepsilon_M{}^N \partial_N g$ as in (4.1). Here $\eta(\xi)$ and $g(\xi)$ are two smooth functions on Σ satisfying $\partial_n \eta = \partial_n g = 0$, and ϵ_{MN} is the volume one form associated to q_{MN}. The transformed Hajicek one-form Ω'_M is then determined by the functions $\eta' = \eta + A$, and $g' = g$. With the change of variables $B = \mathrm{e}^{-\eta'}\tilde{B}$ we find

$$B_M = \tilde{B}_M \mathrm{e}^{-\eta'} - \eta'_M \tilde{B} \mathrm{e}^{-\eta'},$$
$$D^M B_M = D^M \tilde{B}_M \mathrm{e}^{-\eta'} - 2\eta'_M \tilde{B}^M \mathrm{e}^{-\eta'} + \eta'^M \eta'_M \tilde{B} \mathrm{e}^{-\eta'} - D^M \eta'_M \tilde{B} \mathrm{e}^{-\eta'},$$

which allows us to rewrite (B.8) as

$$D^M \tilde{B}_M + 2\varepsilon_M{}^N \tilde{B}^M g_N + \tilde{B}(g^M g_M - \theta^0) - (\theta^0 - \frac{1}{2}\mathcal{R})\mathrm{e}^{\eta'} = 0. \tag{B.9}$$

In this form we can easily identify two solutions to this equation

$$\tilde{B} = -1, \;\; A = \log\left(\frac{\theta_0 - g^M g_M}{\theta^0 - \frac{1}{2}\mathcal{R}}\right) - \eta, \tag{B.10}$$

and

$$\tilde{B} = 1, \;\; A = \log\left(\frac{g^M g_M - \theta^0}{\theta^0 - \frac{1}{2}\mathcal{R}}\right) - \eta, \tag{B.11}$$

provided the argument of the logarithms in these equations are non-negative, and that (B.2) is well defined at $\xi^1 = 0$. We find the following possibilities

$$\tilde{B} = -1: \quad g^M g_M \geqslant \frac{1}{2}\mathcal{R} \geqslant \theta^0, \quad \text{or} \quad g^M g_M \leqslant \frac{1}{2}\mathcal{R} \leqslant \theta^0, \tag{B.12}$$

$$\tilde{B} = 1: \quad g^M g_M \leqslant \theta^0 \leqslant \frac{1}{2}\mathcal{R}, \quad \text{or} \quad g^M g_M \geqslant \theta^0 \geqslant \frac{1}{2}\mathcal{R}. \tag{B.13}$$

Thus, each of these four sets of conditions (to be met at the spatial slice $\mathcal{S}_{\xi^1=0}$) is sufficient to ensure that the gauge $\partial_n \theta^0 = 0$ exists.

B.1.2. Uniqueness of the gauge. Now we will show that this gauge is unique up to supertranslations provided $g^M g_M \leqslant \frac{1}{2}\mathcal{R}$ on Σ. For this, suppose that we are already in this gauge. We would like to know what the possible transformations which maintain this gauge are. They would have to solve (B.9) with $\theta^0 = \frac{1}{2}\mathcal{R}$,

$$D^M \tilde{B}_M + 2\epsilon^{MN} \tilde{B}_M g_N + (g^M g_M - \frac{1}{2}\mathcal{R})\tilde{B} = 0. \tag{B.14}$$

We can multiply by $\tilde{B}$ and integrate over the sphere. After integrating the first two terms by parts and dropping the boundary terms (the spatial sections of $\mathcal{S}_{\xi^1} \cong \mathbb{S}^2$ are compact and simply connected) we get

$$\int_{S^2} d^2\xi \left(\tilde{B}^M \tilde{B}_M + (\frac{1}{2}\mathcal{R} - g^M g_M)\tilde{B}^2\right) = 0. \tag{B.15}$$

This implies that, if

$$\frac{1}{2}\mathcal{R} \geqslant g^M g_M \tag{B.16}$$

is satisfied the integral is the sum of two positive contributions, so we must have $B = \tilde{B} = 0$. But A is unconstrained, so the condition $\partial_n \theta^0 = 0$ fixes the gauge up to transformations for which $f(\xi) = \xi^1 + A(\xi^M)$, i.e. supertranslations. Then we arrive to the following conditions which guarantee that the gauge (3.11) can be reduced down to supertranslations

$$\tilde{B} = -1: \quad g^M g_M \leqslant \frac{1}{2}\mathcal{R} \leqslant \theta^0, \tag{B.17}$$

$$\tilde{B} = 1: \quad g^M g_M \leqslant \theta^0 \leqslant \frac{1}{2}\mathcal{R}. \tag{B.18}$$

Noting that $g^M g_M = \Omega^{0M}\Omega^0_M$ we arrive to (B.1).

A priori it might seem that the first situation, for which $B < 0$, is ill-behaved because the domain of $\zeta(\xi)$ covers only the range

$$\xi^1 \in \big[-\frac{1}{\kappa_0}\log(-1/B), \infty\big), \tag{B.19}$$

but this is not the case. Actually, the consistency condition that we should impose on $\zeta(\xi)$ is that given a tensor field defined on Σ, e.g. γ_{ab}, the coordinate representation of the transformed tensor $\zeta^*\gamma(\xi)$ contains the same information as the original one $\gamma(\xi)$. Thus, we must require that the image of ζ is the full abstract manifold Σ. This condition ensures that scanning over the domain of $\zeta^*\gamma(\xi)$ we will access the full domain where $\gamma(\xi)$ is defined. It is straightforward to see that the image of ζ for the two cases in (B.1) covers the following ranges of the null coordinate

$$(i)\ B < 0: \quad \hat{f}(\xi) \in (-\infty, \infty), \tag{B.20}$$

$$(ii)\ B > 0: \quad \hat{f}(\xi) \in (A + \frac{1}{\kappa_0}\log B, \infty). \tag{B.21}$$

Then, only diffeomorphisms satisfying the condition (i) are well-behaved in the sense explained above. This is the condition we presented in the main text (4.12).

B.1.3. Supertranslation independent condition. We will now prove that the condition (i) in (B.1) is preserved by supertranslations, even before setting the gauge $\partial_n\theta^0 = 0$. To see this, first note that the condition (B.16) is preserved by supertranslations since both $\mathcal{R}$ and g_M transform as scalar fields under supertranslations, but neither of the two depend on the null coordinate, so they are actually invariant. Finally, θ^0 also transforms as a scalar under supertranslations $\theta^0(\xi) \to \theta^{0\prime}(\xi) = \theta^0(\zeta(\xi))$, as it can be checked setting $B = 0$ in (B.7). However, due to the form of (4.9), the rhs cannot change sign, implying that if for some ξ^1 the inequality $\theta^0 \geqslant \frac{1}{2}\mathcal{R}$ holds, then it will hold for all ξ^1, including the supertranslated one $\hat{f}(\xi) = \xi^1 + A(\xi^M)$. Therefore, it is impossible to cross the bound $\theta^0 \geqslant \frac{1}{2}\mathcal{R}$ with a supertranslation.

In conclusion, we have shown that if a given data set satisfies the condition (i) in (B.1), then there is always a gauge transformation of the form (B.2) which allows us to set $\partial_n\theta^0 = 0$ everywhere on the horizon. Furthermore, the gauge freedom that remains once we have done so is that of supertranslations.

B.2. Weyl scalars

In the present section we will compute the Weyl scalars Ψ_n, with $n = \{0,1,2,3\}$, at a generic point $\xi^a = \xi_0^a$ of a non-expanding horizon which is embedded in vacuum, i.e. $T_{\mu\nu} = 0$. As explained in the main text, the scalar Ψ_4 involves information about the spacetime connection off the hypersurface, and thus it cannot be computed from the connection coefficients (2.7).

To be more precise, we will compute the pullback of Ψ_n to the abstract manifold Σ, but we will keep the pullback operation implicit in order to ease the notation. We will use the setting described in section 2.3: we choose a coordinate system for the abstract manifold Σ such that $q_{MN}(\xi_0) = \delta_{MN}$, and we will define the Weyl scalars in terms of the Newman–Penrose tetrad $\mathcal{B}_{NP} = \{n, \ell, m, \overline{m}\}$, where m and $\overline{m}$ are defined by (2.19). The Weyl scalars are given by the expressions

$$
\begin{aligned}
\Psi_0 &= C_{\mu\nu\rho\sigma} n^\mu m^\nu n^\rho m^\sigma = \frac{1}{2}(C_{n2n2} - C_{n3n3}) + \mathrm{i}C_{n2n3},\\
\Psi_1 &= C_{\mu\nu\rho\sigma} n^\mu m^\nu \ell^\rho n^\sigma = \frac{1}{\sqrt{2}}(C_{n2\ell n} + \mathrm{i}C_{n3\ell n}),\\
\Psi_2 &= C_{\mu\nu\rho\sigma} n^\mu m^\nu \ell^\rho \overline{m}^\sigma = \frac{1}{2}(C_{\ell 2n2} + C_{\ell 3n3}) + \frac{\mathrm{i}}{2}(C_{\ell 2n3} - C_{\ell 3n2}),\\
\Psi_3 &= C_{\mu\nu\rho\sigma} n^\mu \ell^\nu \overline{m}^\rho \ell^\sigma = \frac{1}{\sqrt{2}}(C_{n\ell 2\ell} - \mathrm{i}C_{n\ell 3\ell}).
\end{aligned}
\tag{B.22}
$$

First we will express the scalars Ψ_n in terms of the connection coefficients, and then we will compute the gauge corrected Weyl scalars, (4.27) and (4.28), which we introduced in section 4.3.

Lemma B.1. *Let $\mathscr{D} = \{q_{MN}, \kappa, \Omega_M, \Xi_{MN}\}$ be the hypersurface data of a non-expanding horizon $\mathcal{H}$ embedded in the vacuum, and let Ψ_n, $n = 0, 1, 2, 3$, be the Weyl scalars defined with respect to the Newman–Penrose tetrad $\mathcal{B}_{NP} = \{n, \ell, m, \overline{m}\}$ on $\mathcal{H}$. Then, the following equations hold*

$$
\Psi_0 = \Psi_1 = 0, \qquad \Psi_2 = -\frac{1}{4}\mathcal{R} + \frac{\mathrm{i}}{2}\mathcal{J},
\tag{B.23}
$$

and

$$
\begin{aligned}
\mathrm{Re}\Psi_3 &= \frac{1}{\sqrt{2}}(D_{[M}\Xi_{2]N} + \Omega_{[M}\Xi_{2]N})q^{MN},\\
\mathrm{Im}\Psi_3 &= -\frac{1}{\sqrt{2}}(D_{[M}\Xi_{3]N} + \Omega_{[M}\Xi_{3]N})q^{MN},
\end{aligned}
\tag{B.24}
$$

where $D_{[M}\Omega_{N]} = \epsilon_{MN}\mathcal{J}$, $\mathcal{R}$ is the Ricci scalar of q_{MN} and ϵ_{MN} the volume form.

Proof. Due to the Einstein's equations, the Ricci tensor vanishes in vacuum $R_{\mu\nu} = 0$, and the Weyl tensor is equal to the Riemann curvature tensor $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$. Taking this into account we can compute Ψ_0 and Ψ_1 using the identities (A.5) and (A.13)

$$
\mathrm{Re}(\Psi_0) = \frac{1}{2}\Big(-\partial_n(\Theta_{22} - \Theta_{33}) + \kappa(\Theta_{22} - \Theta_{33}) + (\Theta_{22}^2 - \Theta_{33}^2)\Big),
\tag{B.25}
$$

$$
\mathrm{Im}(\Psi_0) = -\partial_n\Theta_{23} + \kappa\Theta_{23} + \Theta_2^C\Theta_{C3},
\tag{B.26}
$$

$$\mathrm{Re}(\Psi_1) = \frac{1}{\sqrt{2}}(\partial_n\Omega_2 - \partial_2\kappa + \Theta_2^N\Omega_N), \tag{B.27}$$

$$\mathrm{Im}(\Psi_1) = \frac{1}{\sqrt{2}}(\partial_n\Omega_3 - \partial_3\kappa + \Theta_3^N\Omega_N). \tag{B.28}$$

Note that, for non-expanding horizons embedded in vacuum all these quantities vanish, $\Psi_0 = \Psi_1 = 0$, since the second fundamental form is zero $\Theta_{MN} = 0$ (see section 3), and as a consequence of the Damour–Navier–Stokes equation (2.13). This proves the left equation in (B.23). It is worth mentioning that this result could also have been obtained using the Goldberg–Sachs theorem (see [66]), and noting the existence of a geodesic and shear free null vector, namely the null normal n.

To compute Ψ_2 we can use the following two identities

$$\begin{aligned} R_{\ell NnM}q^{NM} &= (-\partial_n\Xi_{MN} - D_M\Omega_N - \Omega_M\Omega_N - \kappa\Xi_{MN} + \Theta_N^P\Xi_{MP})q^{NM}, \\ R_{\ell[NnM]} &= D_{[N}\Omega_{M]} - \Xi^L_{[N}\Theta_{M]L}. \end{aligned} \tag{B.29}$$

The first one follows from (A.17), and the second one from the Ricci identity $R_{\ell 2n3} = \ell_\rho n^\mu e_3^\nu \nabla_{[\mu}\nabla_{\nu]}e_2^\rho$. From them we obtain

$$\mathrm{Re}(\Psi_2) = \frac{1}{2}(-\partial_n\theta^\ell - D_M\Omega^M - \kappa\theta^\ell - \Omega^A\Omega_A + \Theta^{AB}\Xi_{AB}), \qquad \mathrm{Im}(\Psi_2) = \frac{1}{2}D_{[2}\Omega_{3]}. \tag{B.30}$$

Here we have also used the fact that the real part of Ψ_2 can also be written as $\mathrm{Re}\Psi_2 = \frac{1}{2}q^{AB}C_{\ell AnB}$. The expressions (4.27) can be recovered when we impose the constraint equations of a non-expanding horizon embedded in vacuum. Setting $\Theta_{MN} = 0$, and from the constraint equation for the trace of Ξ_{MN}, (2.14) we arrive to

$$\mathrm{Re}(\Psi_2) = -\frac{1}{4}\mathcal{R}, \qquad \mathrm{Im}(\Psi_2) = \frac{1}{2}D_{[2}\Omega_{3]}. \tag{B.31}$$

At the point where we are evaluating the expressions, ξ_0^a, the spatial metric has the canonical form $q_{MN} = \delta_{MN}$, and therefore the volume form reduces to the Levi-Civita symbol, which satisfies $\epsilon_{23} = 1$. Thus $D_{[2}\Omega_{3]} \equiv \epsilon_{23}\mathcal{J} = \mathcal{J}$, which proves (B.23).

To compute the last Weyl scalar Ψ_3 it is convenient to use the equation $R_{n\ell A\ell} = R_{\ell MAN}q^{MN}$ which holds in vacuum. It follows from

$$0 = R_{\ell A} = R_{\ell\mu A\nu}g^{\mu\nu} = R_{\ell MAN}q^{MN} + R_{\ell nA\ell} + R_{\ell\ell An} = R_{\ell MAN}q^{MN} - R_{n\ell A\ell} \tag{B.32}$$

with $M \neq A$. Here we used the form for the inverse metric (2.6), and the symmetries of the Riemann tensor, which imply $R_{\ell\ell An} = 0$. Then, the contractions of the Riemann curvature of the form $R_{\ell MAN}$ can be calculated from the relation

$$R_{\ell MAN} = D_{[N}\Xi_{A]M} + \Omega_{[N}\Xi_{A]M}, \tag{B.33}$$

which is a direct consequence of the Ricci identity $R_{\ell MAN} = \ell_\sigma e_N^\mu e_M^\nu \nabla_{[\mu}\nabla_{\nu]}e_A^\sigma$, and the definitions of the connection coefficients (2.7). Recalling that the Riemann and Weyl tensors are equal in vacuum we have that $C_{n\ell A\ell} = R_{\ell MAN}q^{MN}$, we can obtain (B.24) using (B.33) and the definitions (B.22). ■

Recall, that the gauge corrected Weyl scalars are given by

$$\Psi_n^c(\eta, \xi^M) \equiv \Psi_n^c(H(\eta, \xi^M), \xi^M),$$

where $H(\eta, \xi^M)$ is defined in (4.18). Therefore, since the Weyl scalars Ψ_0, Ψ_1 and Ψ_2 do not depend on the null coordinate ξ^1, their gauge corrected expressions are identical to those in (B.23), which proves (4.27).

It only remains to derive the expression (4.28) for the gauge corrected Weyl scalar $\Psi_3^c(\eta, \xi^M)$.

Proposition B.1. *Let $\mathscr{D} = \{q_{MN}, \kappa, \Omega_M, \Xi_{MN}\}$ represent the hypersurface data of a generic non-expanding horizon embedded in vacuum, and let $\Psi_3^c(\eta, \xi^M)$ be the gauge corrected Weyl scalar defined in (4.26). Then, in the gauge defined by (4.6) and (4.11), $\Psi_3^c(\eta, \xi^M)$ is given by (4.28).*

Proof. We will first write Ψ_3 in terms of the object Σ^0_{MN} defined (4.4). Substituting the definition (4.4) into (B.33), after a straightforward calculation we find

$$\kappa C_{n\ell A\ell} = (D_{[N}\Sigma^0_{A]M} + \Omega_{[N}\Sigma^0_{A]M})q^{MN} + \frac{1}{2}\epsilon_{AM}D^M\mathcal{J} + \frac{3}{2}\epsilon_{AC}\Omega^C\mathcal{J} - \frac{1}{2}\Omega_A\mathcal{R}, \tag{B.34}$$

where $\mathcal{R}$ and ϵ_{MN} are, respectively, the curvature scalar and volume form of q_{MN}, and $\mathcal{J} = D_{[2}\Omega_{3]}$. In order to simplify this expression we can use the assumption that the horizon is generic, and that the gauge redundancies (3.11) have been partially fixed by the conventions (4.6) and (4.11). Then, the trace of Σ^0_{MN} satisfies $\theta^0 = \frac{1}{2}\mathcal{R}$, and thus the first term in the previous equation takes the form

$$(D_{[N}\Sigma^0_{A]M} + \Omega_{[N}\Sigma^0_{A]M})q^{NM} = D^M\sigma^0_{AM} + \Omega^M\sigma^0_{AM} - \frac{1}{4}\partial_A\mathcal{R} - \frac{1}{4}\Omega_A\mathcal{R}, \tag{B.35}$$

where σ^0_{MN} is the traceless part of Σ^0_{MN}. This leads to

$$\kappa C_{n\ell A\ell} = D^M\sigma^0_{AM} + \Omega^M\sigma^0_{AM} - \frac{1}{4}\partial_A\mathcal{R} + \frac{1}{2}\epsilon_{AM}D^M\mathcal{J} + \frac{3}{2}\epsilon_{AC}\Omega^C\mathcal{J} - \frac{3}{4}\Omega_A\mathcal{R}. \tag{B.36}$$

Then from the definition of Ψ_3 we find

$$\begin{aligned}
\mathrm{Re}\Psi_3 &= \frac{1}{\kappa\sqrt{2}}(D^M\sigma^0_{2M} + \Omega^M\sigma^0_{2M} - \frac{1}{4}\partial_2\mathcal{R} + \frac{1}{2}D_3\mathcal{J} + \frac{3}{2}\Omega_3\mathcal{J} - \frac{3}{4}\Omega_2\mathcal{R})\\
\mathrm{Im}\Psi_3 &= \frac{1}{\kappa\sqrt{2}}(-D^M\sigma^0_{3M} - \Omega^M\sigma^0_{3M} + \frac{1}{4}\partial_3\mathcal{R} + \frac{1}{2}D_2\mathcal{J} + \frac{3}{2}\Omega_2\mathcal{J} + \frac{3}{4}\Omega_3\mathcal{R}),
\end{aligned} \tag{B.37}$$

or equivalently

$$\Psi_3 = \frac{1}{\kappa\sqrt{2}}(D\sigma^0 + \hat{D}\Psi_2 + 3\,\hat{\Omega}\,\Psi_2), \tag{B.38}$$

where we used the shorthands $\hat{D} \equiv D_2 - \mathrm{i}D_3$ and $\hat{\Omega} \equiv \Omega_2 - \mathrm{i}\Omega_3$, and we defined the complex scalar

$$D\sigma^0 \equiv D^M\sigma^0_{2M} + \Omega^M\sigma^0_{2M} - \mathrm{i}(D^M\sigma^0_{3M} + \Omega^M\sigma^0_{3M}). \tag{B.39}$$

Then, the gauge corrected Weyl scalar is given by

$$\begin{aligned}\Psi_3^c(\eta,\xi^M) &= \frac{1}{\kappa\sqrt{2}}(D\sigma^0 + \hat{D}\Psi_2 + 3\,\hat{\Omega}\,\Psi_2)|_{\xi^1=H(\eta,\xi^M)} \\ &= \frac{1}{\kappa\sqrt{2}}D\sigma^0|_{\xi^1=H(\eta,\xi^M)} + \frac{1}{\kappa\sqrt{2}}(\hat{D}\Psi_2^c + 3\,\hat{\Omega}\,\Psi_2^c),\end{aligned} \tag{B.40}$$

where the second equality follows from the fact that neither Ψ_2, Ω_M, or q_{MN} depend on ξ^1, and thus their functional form is unchanged after evaluating them in $\xi^1 = H(\eta,\xi^M)$. Therefore we just have to compute the first term on the right in the last equation, which reads

$$\begin{aligned}D\sigma^0|_{\xi^1=H(\eta,\xi^M)} = D^M\sigma^0_{2M}|_{\xi^1=H(\eta,\xi^M)} + \Omega^M\sigma_{2M} \\ - \mathrm{i}(D^M\sigma^0_{3M}|_{\xi^1=H(\eta,\xi^M)} + \Omega^M\sigma_{3M}).\end{aligned} \tag{B.41}$$

In the previous expression we already made the substitution $\sigma^0_{MN}|_{\xi^1=H(\eta,\xi^M)} = \sigma_{MN}$, using the definition of the supertranslation invariant variable σ_{MN}, i.e. (4.20). From the definition (4.20) it is essential to check that the following relation holds

$$\begin{aligned}D^M\sigma_{2M} &= (D^M\sigma^0_{2M} + \partial_n\sigma^0_{2M}\,\partial^M H)|_{\xi^1=H(\eta,\xi^M)} \\ &= (D^M\sigma^0_{2M} + \sigma^0_{2M}\,\Omega^{e|M})|_{\xi^1=H(\eta,\xi^M)} \\ &= D^M\sigma^0_{2M}|_{\xi^1=H(\eta,\xi^M)} + \sigma_{2M}\,\Omega^{e|M}\end{aligned} \tag{B.42}$$

where, for the second equality, we have used the equations (4.16) and (4.18), and that the exact part of the Hajicek one form is given by $\partial_M\eta = \Omega^e_M$. The last result allows us to express Ψ_3^c in terms of the gauge invariant variable σ_{MN} as follows

$$\Psi_3^c(\eta,\xi^M) = \frac{1}{\kappa\sqrt{2}}(D\sigma + \hat{D}\Psi_2^c + 3\,\hat{\Omega}\,\Psi_2^c), \tag{B.43}$$

where $D\sigma$ is now is defined as

$$D\sigma(\eta,\xi^M) \equiv D^M\sigma_{2M} + \Omega^{0|M}\sigma_{2M} - \mathrm{i}(D^M\sigma_{3M} + \Omega^{0|M}\sigma_{3M}). \tag{B.44}$$

It can be seen that the exact part of the Hajicek one form Ω^e_M has been cancelled out, and thus $D\sigma$ only involves the divergence free part Ω^0_M. Substituting the solution to the constraint equations (4.23) in (B.43) we arrive to our final result, which is given by (4.28). ■

Appendix C. Calculations for null infinity

C.1. Derivation of the BMS group

In this appendix we present a derivation of the BMS group of transformations at null infinity. We show that it can be described as the set of diffeomorphisms of the unphysical spacetime which preserve null infinity as a set of points, and that leave invariant both the metric tensor and the null normal on I up to a conformal transformation. The analysis is done in a similar fashion as in section 3.3, where we studied the set of diffeomorphisms preserving the metric tensor at a non-expanding horizon, i.e. the hypersurface symmetries, which include horizon supertranslations. Therefore, the present computation also serves as a check for our approach in section 3.3 to characterise horizon supertranslations.

We will now characterise the set of diffeomorphisms of the unphysical spacetime $F:\mathcal{M}\rightarrow\mathcal{M}$ which preserve the structure of null infinity implied by the definition of asymptotically flat spacetimes given in section 5, i.e. conditions (i), (ii) and (iii). More specifically, any diffeomorphism F in this set should satisfy the following conditions:

(i) Leave invariant the scalar products at $\mathcal{I}$ up to a conformal transformation. That is, denoting $g'_{\mu\nu}\equiv(F^{*}g)_{\mu\nu}$, and $\Omega'\equiv F^{*}\Omega$ we should have

$$g'_{\mu\nu}\hat{=}\,\omega^{2}g_{\mu\nu},\quad\text{and}\quad\Omega'\hat{=}\,\omega\Omega,\tag{C.1}$$

so that $\Omega'^{-2}g'_{\mu\nu}\hat{=}\,\hat{g}_{\mu\nu}$, where $\hat{=}$ denotes equality on $\mathcal{I}$.

(ii) Map null infinity to itself, $F(\mathcal{I})=\mathcal{I}$, or equivalently $F^{*}\Omega\,\hat{=}\,\Omega\,\hat{=}\,0$.

(iii) Preserve the definition of the null normal, $\boldsymbol{n}\equiv\mathrm{d}\Omega$.

The properties of this set of diffeomorphisms are more easily studied using a coordinate system of the unphysical spacetime adapted to $\mathcal{I}$. We proceed as in section 3.3 for the case of non-expanding horizons. Since null infinity I is diffeomorphically identified with the abstract manifold $\mathcal{I}$ via the embedding map Φ, we can use the coordinates on the later one, ξ^{a}, to parametrise the hypersurface I. The coordinate system on I is then extended off the hypersurface introducing a transverse coordinate r, which is defined in terms of the rigging as $\ell=\partial_{r}$, with $r(\mathcal{I})=0$, and then keeping the coordinates ξ^{a} constant along the integral curves of ℓ. Thus, the coordinate system for the unphysical spacetime reads $x^{\mu}=\{\xi^{1},r,\xi^{M}\}$, so that the embedding map takes the simple form

$$\Phi:\xi^{a}\longrightarrow x^{\mu}=\{u=\xi^{1},r=0,x^{M}=\xi^{M}\}.\tag{C.2}$$

The elements of coordinate basis $\mathcal{B}=\{n,\ell,e_{M}\}$ have the following explicit form

$$n=\partial_{0},\qquad\ell=\partial_{1},\qquad e_{M}=\partial_{M}.\tag{C.3}$$

Then the null normal has the coordinate form $\boldsymbol{n}=dr$, what follows from our conventions in section 2.1, $\boldsymbol{n}(\ell)=1$, and the properties of the null normal $\boldsymbol{n}(e_{M})=\boldsymbol{n}(n)=0$. Moreover, on this coordinate system the metric tensor has the following form at $\mathcal{I}$

$$g_{\mu\nu}|_{\mathcal{I}}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&q_{MN}\end{pmatrix}.\tag{C.4}$$

Let us turn to the characterisation of the properties of the diffeomorphisms F satisfying the conditions (i), (ii) and (iii) above. From the condition (iii) it is immediate to find the required behaviour of the null normal under the pull back F^{*}. Indeed, since $\Omega\,\hat{=}\,0$, we have

$$\mathrm{d}\Omega'\,\hat{=}\,(\omega\mathrm{d}\Omega+\Omega\mathrm{d}\omega)\,\hat{=}\,\omega\mathrm{d}\Omega\qquad\Longrightarrow\qquad\boldsymbol{n}'\,\hat{=}\,\omega\boldsymbol{n}\tag{C.5}$$

where $\boldsymbol{n}'\equiv F^{*}\boldsymbol{n}$. From this we can also find the transformation of the normal vector $n=g^{-1}(\boldsymbol{n},\cdot)$ under the pushforward of F. On the one hand, from the definition of n we have that for any $k\in T_{p}\mathcal{I}$

$$F^{*}g(n,k)\,\hat{=}\,\omega^{2}g(n,k)\,\hat{=}\,\omega^{2}\boldsymbol{n}(k).\tag{C.6}$$

On the other hand, since F maps $\mathcal{I}$ to itself, it follows that n can change at most by a rescaling $dF(n)=\alpha n$. Then

$$\begin{aligned}F^{*}g(n,k)&\,\hat{=}\,g(\mathrm{d}F(n),\mathrm{d}F(k))\,\hat{=}\,\alpha g(n,\mathrm{d}F(k))\\&\,\hat{=}\,\alpha\boldsymbol{n}(\mathrm{d}F(k))\,\hat{=}\,\alpha F^{*}\boldsymbol{n}(k)\,\hat{=}\,\alpha\omega\boldsymbol{n}(k).\end{aligned}\tag{C.7}$$

Comparing the two previous expressions we find $\alpha = \omega$. Let $y^\alpha = y^\alpha(x)$ be the explicit form for the diffeomorphism F, in a given set of coordinates, then from the condition on the pull-back of $\boldsymbol{n}$ (C.6) we find the following constraints on the mapping F

$$F^*\boldsymbol{n}_\mu \hat{=} y^\alpha_\mu \boldsymbol{n}_\alpha \qquad \Longrightarrow \qquad y^1_\mu \hat{=} \omega\delta^1_\mu, \tag{C.8}$$

where we use the short hands $y^\alpha_\mu = \partial_\mu y^\alpha$. From the condition on the pushforward of n (C.7) we find

$$\mathrm{d}F(n)^\alpha \hat{=} y^\alpha_\mu n^\mu \qquad \Longrightarrow \qquad y^\alpha_0 \hat{=} \omega\delta^\alpha_0. \tag{C.9}$$

Collecting both results we have

$$y^0_0 \hat{=} y^1_1 \hat{=} \omega, \qquad y^1_0 \hat{=} y^1_M \hat{=} y^I_0 \hat{=} 0. \tag{C.10}$$

The mapping F satisfies conditions (C.1), if and only if the scalar products on the basis $\mathcal{B}$ are mapped as follows

$$(F^*g)(n,n) \hat{=} 0, \qquad (F^*g)(e_M,e_N) \hat{=} \omega^2 q_{MN} \tag{C.11}$$

$$(F^*g)(n,e_M) \hat{=} 0, \qquad (F^*g)(e_M,\ell) \hat{=} 0 \tag{C.12}$$

$$(F^*g)(n,\ell) \hat{=} \omega^2, \qquad (F^*g)(\ell,\ell) \hat{=} 0. \tag{C.13}$$

In components they read

$$g_{\alpha\beta}y^\alpha_1 y^\beta_1 \hat{=} g_{11}\omega^2 \hat{=} 0 \qquad g_{\alpha\beta}y^\alpha_M y^\beta_N \hat{=} q_{IJ}y^I_M y^J_N \hat{=} \omega^2 q_{MN} \tag{C.14}$$

$$g_{\alpha\beta}y^\alpha_0 y^\beta_M \hat{=} \omega\, y^1_M \hat{=} 0 \qquad g_{\alpha\beta}y^\alpha_M y^\beta_1 \hat{=} y^0_M\omega + q_{IJ}y^I_1 y^J_M \hat{=} 0 \tag{C.15}$$

$$g_{\alpha\beta}y^\alpha_0 y^\beta_1 \hat{=} \omega^2 \qquad g_{\alpha\beta}y^\alpha_1 y^\beta_1 \hat{=} 2y^0_1\omega + q_{IJ}y^I_1 y^J_1 \hat{=} 0. \tag{C.16}$$

The second equation of the first line implies that, on $\mathcal{I}$, $Y^I(x^M) \equiv y^I(x^M)|_{r=0}$ define a conformal symmetry of the metric q_{MN} with conformal factor ω,

$$\boxed{q_{IJ}Y^I_M Y^J_N = \omega^2 q_{MN}.} \tag{C.17}$$

Note that, since Y^I are constant along the null coordinate u, the conformal factor must satisfy $\mathcal{L}_n\omega \hat{=} 0$. If we restrict ourselves to globally well-defined transformations, that is, to one-to-one mappings of the spatial sections of $\mathcal{I}$ on to themselves, then the functions Y^I generate a group isomorphic to the homogeneous orthochronous Lorentz group (see [23]).

The action of the diffeomorphism on the null coordinate at $\mathcal{I}$ is determined by the function $f(u,x^M) \equiv y^0(u,x^M)|_{r=0}$, which is constrained by the first equality in (C.10), namely $y^0_0 \hat{=} \omega$. Thus, the function $f(u,x^M)$ has the general form

$$\boxed{f(u,x^M) = \omega(x^M)\big(u + A(x^M)\big),} \tag{C.18}$$

where $A(x^M)$ can be any smooth function of the spatial coordinates x^M. The remaining non-trivial conditions can be solved in terms of f and Y^I to give

$$y^I_1 \hat{=} -\frac{1}{\omega}f_M q^{MN} Y^I_N, \qquad y^0_1 \hat{=} -\frac{1}{2\omega}f_M f^M. \tag{C.19}$$

The set of transformations determined by the functions (f, Y^I) given by (C.17) and (C.18) define the BMS group (see e.g. chapter 1 in [65]). *Null infinity supertranslations* can be identified as those transformations of the BMS group with $\omega \hat{=} 1$, and $Y^I(x) = x^I$, that is

$$\boxed{f(u, x^M) = x^0 + A(x^M).} \tag{C.20}$$

The infinitesimal version of the defining conditions of BMS transformations can be recovered setting $y^\alpha(x) \approx x^\alpha + \epsilon k^\alpha(x)$, in (C.1) and (C.5), where the vector field k^α is the corresponding generator and $\epsilon \ll 1$ is a small real parameter. We obtain

$$\mathcal{L}_k g_{\mu\nu} = 2\lambda g_{\mu\nu}, \qquad \mathcal{L}_k n = -\lambda n, \tag{C.21}$$

where $\omega \approx 1 + \lambda$, and $\mathcal{L}_n \lambda = 0$. This is precisely the definition used to characterised the BMS group in the works by Geroch and Ashtekar [26, 52] (see also [54]). For the second equality we have used that the definition of the Lie derivative of a vector field involves the pushforward of the inverse mapping F^{-1}, this explains the minus sign on the second expression.

From the previous equations it is also straightforward to find the action of a supertranslation on the tensor Y_{ab} at null infinity. Indeed, if we adopt the gauge conventions in section 5, null infinity can be described as a non-expanding null hypersurface. Moreover, since supertranslations preserve exactly the metric tensor and the null normal at null infinity they can be identified with a hypersurface symmetry of $\mathcal{I}$. Therefore, we can use the results in section 3.3 to find the transformation properties of the tensor Y_{ab} under null infinity supertranslations, which hold for arbitrary hypersurface symmetries of a generic non-expanding null hypersurface. Setting $\kappa = \Omega = 0$ and $\hat{f}_1 = 1$ on equation (3.30) we obtain

$$Y'_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \Xi_{MN} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & D_M A_N \end{pmatrix}, \tag{C.22}$$

which is the relation we have used in the main text (5.4).

C.2. Boundary conditions for no-outgoing radiation

In this appendix we will derive the equations (5.12)–(5.14), that is, the constraints satisfied by the Schouten tensor $S_{\mu\nu}$ of the unphysical spacetime in the absence of outgoing gravitational radiation at null infinity. Our starting point are the boundary conditions (5.10) expressed in terms of the leading order Weyl scalars Ψ^0_n, and the Bianchi identities (5.11), relating the leading order unphysical Weyl tensor $K_{\mu\nu\rho\sigma}$ with $S_{\mu\nu}$.

We prepare our set up as described in section 2.3. Given a point ξ^a_0 at null infinity, the Weyl scalars Ψ^0_n can be expressed in terms of the Newman–Penrose null $\mathcal{B}_{NP} = \{\ell, n, m, \overline{m}\}$, where ℓ is the rigging vector, n the null normal vector to $\mathcal{I}$, and the complex null vectors m and $\overline{m}$ are defined as in (2.19). In addition we choose the coordinates on $\mathcal{I}$ such that, at the point $\xi^a = \xi^a_0$, the spatial metric has the canonical form $q_{MN} = \delta_{MN}$.

C.2.1. Condition on the second Weyl scalar. We begin deriving the constraint on $S_{\mu\nu}$ which follows from imposing $\mathrm{Im}\Psi^0_2 = 0$ at null infinity. The second Weyl scalar Ψ^0_2 has the form

$$\Psi^0_2 = K_{\ell m n \overline{m}}, \quad \Longrightarrow \quad \mathrm{Im}\Psi^0_2 = \frac{1}{2}(K_{\ell 3 n 2} - K_{\ell 2 n 3}). \tag{C.23}$$

Here we will use the notation $K_{\ell m n \overline{m}} = K^\mu_{\ \nu\rho\sigma}\ell_\mu m^\nu n^\rho \overline{m}^\sigma$, and similar expressions for contractions of a tensor with the elements of a basis. The last term can be rewritten using the symmetries of the Weyl tensor, and the first (algebraic) Bianchi identity

$$\mathrm{Im}\Psi_2^0 = \frac{1}{2}(K_{\ell 3n2} - K_{\ell 2n3}) = \frac{1}{2}(K_{n2\ell 3} + K_{n32\ell}) = -\frac{1}{2}K_{n\ell 32} = \frac{1}{2}K_{32\ell n}. \tag{C.24}$$

Using the Bianchi identity (5.11), together with the definitions for the connection coefficients (2.7), and the gauge conventions (5.1) we find

$$\mathrm{Im}\Psi_2^0 = \frac{1}{2}K_{32\ell n} = -\frac{1}{2}\nabla_{[\mu}S_{\nu]\rho}\ell^\rho e_3^\mu e_2^\nu = -\frac{1}{2}(D_{[3}S_{2]\ell} - \Xi^C_{[3}S_{2]C}), \tag{C.25}$$

where $S_{M\ell} = S_{\mu\nu}e^\mu_M \ell^\nu$ and $S_{MN} = S_{\mu\nu}e^\mu_M e^\nu_N$. It is easy to check that only the traceless part of Ξ_{MN} contributes in the previous equation. The condition $\mathrm{Im}\Psi_2^0 = 0$ implies that the previous expression should vanish, what can be written more covariantly as

$$D_{[M}S_{N]\ell} = \Xi^P_{[M}S_{N]P}. \tag{C.26}$$

C.2.2. Condition on the third Weyl scalar. The vanishing of the third Weyl scalar on $\mathcal{I}$, $\Psi_3^0 = 0$, leads to two constraints on the Schouten tensor. First let us write third Weyl scalar as

$$\Psi_3^0 = K_{\ell n n\overline{m}} = K_{n\overline{m}\ell n} = \frac{1}{\sqrt{2}}(K_{n2\ell n} - \mathrm{i}K_{n3\ell n}). \tag{C.27}$$

Using the Bianchi identity (5.11), together with (2.7) and (5.1), we can rewrite this expression as

$$K_{nM\ell n} = -n^\mu e^\nu_M \nabla_{[\mu}S_{\nu]\rho}\ell^\rho = -(\partial_n S_{M\ell} - \partial_M S_{n\ell}), \tag{C.28}$$

and therefore

$$\Psi_3^0 = -\frac{1}{\sqrt{2}}(\partial_{[n}S_{2]\ell} - \mathrm{i}\partial_{[n}S_{3]\ell}). \tag{C.29}$$

In the absence of outgoing radiation on $\mathcal{I}$ the previous expression vanishes, or equivalently

$$\partial_{[n}S_{M]\ell} = 0. \tag{C.30}$$

The second constraint for the Schouten tensor can be found writing the third Weyl scalar as follows

$$\Psi_3^0 = \frac{1}{\sqrt{2}}(K_{n2\ell n} - \mathrm{i}K_{n3\ell n}) = -\frac{1}{\sqrt{2}}(K_{323n} - \mathrm{i}K_{232n}) = \frac{1}{\sqrt{2}}(D_{[3}S_{2]3} - \mathrm{i}D_{[2}S_{3]2}). \tag{C.31}$$

The second equality is a consequence of the Weyl tensor being traceless, $g^{\mu\nu}K_{\mu 2\nu n} = 0$. In particular, using equation (2.6) for the inverse metric, with $q_{MN} = \delta_{MN}$, and the symmetries of the Weyl tensor we find

$$g^{\mu\nu}K_{\mu 2\nu n} = K_{\ell 2nn} + K_{n2\ell n} + K_{222n} + K_{323n} = K_{n2\ell n} + K_{323n} = 0. \tag{C.32}$$

The last equality in (C.31) is obtained after using the Bianchi identity (5.11), together with (2.7) and (5.1). Thus, the vanishing of the third Weyl scalar also implies the tensor equation

$$D_{[M}S_{N]P} = 0. \tag{C.33}$$

C.2.3. Condition on the fourth Weyl scalar. The last constraint on $S_{\mu\nu}$ is obtained from the vanishing of Ψ_4^0, which reads

$$\Psi_4^0 = K_{\overline{m}n\overline{m}n} = \frac{1}{2}(K_{2n2n} - K_{3n3n}) - \mathrm{i}K_{2n3n}. \tag{C.34}$$

The Bianchi identities (5.11), (2.7) and (5.1) imply $K_{MnNn} = \partial_n S_{MN}$, and thus

$$\Psi_4^0 = \frac{1}{2}(\partial_n S_{22} - \partial_n S_{33}) - \mathrm{i}\partial_n S_{23} = 0. \tag{C.35}$$

In addition, the Schouten tensor satisfies $S_M^M = \mathcal{R}$, which together with $q_{MN} = \delta_{MN}$, implies $\partial_n \mathcal{R} = \partial_n S_{22} + \partial_n S_{33} = 0$. Therefore, we can summarise the constraints which follow from $\Psi_4^0 = 0$ in the tensor equation

$$\partial_n S_{MN} = 0. \tag{C.36}$$

This completes our proof of equations (5.12)–(5.14).

C.3. Constraint equations at null infinity and radiative vacua

In this appendix we prove various results needed in section 5 to derive the constraint equations of null infinity, and to find their solutions in the absence of outgoing radiation.

C.3.1. Consistency check for the constraint equations. In this section we check that the dependence of Ξ_{MN} on the null coordinate ξ^1 implied by the constraint equations (5.19) is consistent with the identity (5.20).

Proposition C.1. *Let Ξ_{MN} be a solution to the constraint equations of null infinity (5.19). Then, if the relation (5.20) is satisfied at a particular value of the null coordinate ξ_0^1, then it will be satisfied for all values of ξ^1.*

Proof. The constraint equation for Ξ_{MN} at null infinity can be obtained from (2.14) substituting the gauge fixing conditions (5.1), and the form of the source term (5.18). The result is

$$\partial_n \Xi_{MN} = -\frac{1}{2}(S_{MN} + S_{n\ell} q_{MN}). \tag{C.37}$$

Note that this equation reduces to (5.19) when we express it in terms of the equivalence relation (5.17). The relation (5.20) is satisfied at a given value of the null coordinate $\xi^1 = \xi_0^1$, they will be satisfied for all values of ξ^1 provided the following expression vanishes on $\mathcal{I}$

$$\partial_n (D_{[M}\Xi_{N]P} - \frac{1}{2} q_{P[M} S_{N]\ell}) = D_{[M}\partial_n \Xi_{N]P} - \frac{1}{2} q_{P[M}\partial_n S_{N]\ell}, \tag{C.38}$$

where the equality is obtained using that the metric q_{MN} and its Levi-Civita connection are independent of ξ^1. Thus, we need to prove that the previous expression is zero. Substituting the constraint equation (C.51) we find

$$D_{[M}\partial_n \Xi_{N]P} - \frac{1}{2} q_{P[M}\partial_n S_{N]\ell} = -\frac{1}{2} D_{[M} S_{N]P} - \frac{1}{2}\partial_{[M} S_{n\ell}\, q_{N]P} - \frac{1}{2} q_{P[M}\partial_n S_{N]\ell}. \tag{C.39}$$

As the indices $M, N, P = \{1,2\}$ and $N \neq M$, then P must be either equal to M or N. Without loss of generality we choose $P = M$. Moreover, in the following we will also assume that we have chosen the coordinates so that $q_{MN} = \delta_{MN}$ locally. We obtain

$$-\frac{1}{2}(D_{[M} S_{N]M} + \partial_{[n} S_{N]\ell}) = \frac{1}{2}(K_{MNMn} + K_{nN\ell n}), \tag{C.40}$$

where we have also used the Bianchi identity (5.11) (contracted with elements of the basis $\mathcal{B} = \{n, \ell, e_M\}$) in the second equality. Using the symmetries of the Weyl tensor, and that $q_{MN} = \delta_{MN}$, we find

$$\frac{1}{2}(K_{MNMn} + K_{nN\ell n}) = \frac{1}{2}(q^{AB}K_{ANBn} + K_{nN\ell n}), \tag{C.41}$$

where A, B run over $\{1, 2\}$. The previous expression can be written in the form

$$\frac{1}{2}(q^{AB}K_{ANBn} + K_{nN\ell n}) = \frac{1}{2}g^{\mu\nu}K_{\mu N\nu n}, \tag{C.42}$$

where we used the formula (2.6) for the inverse metric. Summarising, we have found the relation

$$\partial_n(D_{[M}\Xi_{N]M} - \frac{1}{2}q_{M[M}S_{N]\ell}) = \frac{1}{2}g^{\mu\nu}K_{\mu N\nu n} \tag{C.43}$$

which can be easily checked to be always zero, since the contraction of any two indices of the Weyl tensor is always zero. ■

C.3.2. Schouten tensor at null infinity with no outgoing radiation. We will begin proving the relation (5.15) which is satisfied by the Schouten tensor in the absence of outgoing radiation through $\mathcal{I}$. According to our discussion above, the boundary conditions (5.10) for no outgoing radiation imply the constraints (C.33), (C.36) for $S_{\mu\nu}$. Moreover, as we proved in section 5, in the divergence free conformal gauge (5.1) the Schouten tensor also satisfies $S^M_M = \mathcal{R}$.

Proposition C.2. *The tensor $S_{MN} = \frac{1}{2}\mathcal{R}q_{MN}$ is the unique solution of the constraints (C.33), (C.36) whose trace is given by $S^M_M = \mathcal{R}$.*

Proof. The tensor S_{MN} can be decomposed in its trace and traceless parts as follows

$$S_{MN} = \sigma_{MN} + \frac{1}{2}\mathcal{R}q_{MN}, \tag{C.44}$$

where $\sigma^M_M = 0$. Then S_{MN} is a solution to the constraints (C.33), (C.36) if and only if the traceless part σ_{MN} satisfies

$$\partial_n\sigma_{MN} = 0, \quad D_{[M}\sigma_{N]P} = 0, \quad \sigma_{MN}q^{MN} = 0. \tag{C.45}$$

To complete our proof we just need to use a result by Geroch [52], who showed that the unique solution to these equations is $\sigma_{MN} = 0$. ■

Since our setting is slightly different to that of [52], we will reproduce here the proof of the last statement:

Lemma C.1. *The only solution to the system of equations (C.45) is the trivial one, that is $\sigma_{MN} = 0$.*

Proof. Let σ_{MN} be a tensor satisfying (C.45) and k^M a killing vector[18] of q_{MN}. For convenience we work in a coordinate system such that $q_{MN} = \delta_{MN}$ locally. We will begin proving that

[18] Recall that, in our conformal gauge, q_{MN} represents the geometry of a two dimensional sphere with constant curvature.

the spatial tensor $\lambda_{MN} \equiv D_{[M}(\sigma_{N]P}k^P) = 0$ is vanishing. Due to the antisymmetry of λ_{MN} we already have $\lambda_{11} = \lambda_{22} = 0$ and $\lambda_{12} = -\lambda_{21}$, and thus, it only remains to show that $\lambda_{12} = 0$. Using (C.45) the it an be checked that the components of λ_{MN} satisfy

$$\lambda_{MN} = k^P D_{[M}\sigma_{N]P} + \sigma_{[NP}D_{M]}k^P = \sigma_{[NP}D_{M]}k^P, \tag{C.46}$$

and therefore we also have

$$\begin{aligned}\lambda_{12} &= \sigma_1^1 D_2 k_1 + \sigma_1^2 D_2 k_2 - \sigma_2^2 D_1 k_1 - \sigma_2^2 D_1 k_2 \\ &= \sigma_1^1 D_2 k_1 - \sigma_2^2 D_1 k_2 = (\sigma_1^1 + \sigma_2^2) D_2 k^1 = 0.\end{aligned} \tag{C.47}$$

Here we have used $\sigma_M^M = 0$, and also the following properties of the killing vector k^M

$$D_{(M}k_{N)} = 0 \quad \Longrightarrow \quad D_1k_1 = D_2k_2 = 0, \quad D_1k_2 = -D_2k_1, \tag{C.48}$$

which are a direct consequence of the killing equation. Therefore we have that $D_{[M}(\sigma_{N]P}k^P) = 0$ and, since the spatial sections of $\mathcal{I}$ are topologically equivalent to $\mathbb{S}^2$, this implies that $\sigma_{NP}k^P = \partial_N\alpha$ for some smooth function $\alpha = \alpha(\xi^M)$. Actually, it is straightforward to check that α must be harmonic

$$\Delta\alpha = D^M(\sigma_{MP}k^P) = k^P q^{MN} D_N\sigma_{MP} + \sigma_{MP}D^N k^P = 0. \tag{C.49}$$

The last term vanishes because $D_M k_N$ is antisymmetric and σ_{MN} symmetric, and the first one can be shown to be zero using the second of the equations (C.45) together with $\sigma_{MN} = \sigma_{NM}$ and $\sigma_M^M = 0$

$$k^P q^{MN} D_N \sigma_{MP} = -k^P D_P \sigma_M^M = 0. \tag{C.50}$$

The only harmonic functions on the spatial sections of $\mathcal{I}$ (which are by assumption compact and simply connected) are constants, and thus we have $\sigma_{MN}k^M = \partial_M\alpha = 0$. Finally, since the killing vectors k^M of the sphere span all of the tangent space, we can conclude that $\sigma_{MN} = 0$. ■

C.3.3. Constraint equations in the absence of outgoing radiation at null infinity. We will now prove that, in the absence of outgoing radiation thought $\mathcal{I}$, the general solution to the constraint equations (2.12), (2.13) and (2.14) at null infinity is given by (5.26).

As discussed in section 5, in the conformal gauge (5.1) the only non-trivial constraint equation is the one of the transverse connection Ξ_{MN} (2.14), together with (5.20). The final form of these constraint equations can be found substituting in them the expression for the gauge conditions (5.1), the form of the source term (5.18), and using the fact that in the absence of radiation the Schouten tensor satisfies (5.20) and $S_{a\ell} = \partial_a S_\ell$ (see section 5). The result is

$$\partial_n \Xi_{MN} = -\frac{1}{4}(\mathcal{R} + 2\partial_n S_\ell) q_{MN}, \qquad D_{[M}\Xi_{N]P} = \frac{1}{2} q_{P[M}\partial_{N]} S_\ell. \tag{C.51}$$

In order to eliminate the redundancy associated with supertranslations we specify a fiducial vacuum connection Ξ^0_{MN}, and then characterise a generic vacuum connection Ξ_{MN} by the difference $\Sigma_{MN} \equiv \Xi_{MN} - \Xi^0_{MN}$. It is straightforward to check that this quantity is invariant under supertranslations (5.4). Thus, given a fixed fiducial vacuum Ξ^0_{MN}, we can characterise the full set of radiative vacua at null infinity finding the most general form of Σ_{MN} which is consistent with the constraint equations (C.51).

Theorem C.1. *Let $\Sigma_{MN} \equiv \Xi_{MN} - \Xi^0_{MN}$ be the difference between two transverse connections, Ξ_{MN} and Ξ^0_{MN}, which solve the constraint equations (C.51) in the absence of outgoing radiation through $\mathcal{I}$. Then Σ_{MN} has the general form*

$$\Sigma_{MN} = D_M f_N + \frac{1}{2}\mathcal{R} f q_{MN} - \frac{1}{2}(S_\ell - S^0_\ell) q_{MN}, \tag{C.52}$$

where $f(\xi)$ is smooth function on $\mathcal{I}$ satisfying $\partial_n f = 0$, and the potentials S_ℓ and S^0_ℓ are defined by $S_{a\ell} = \partial_a S_\ell$ and $S^0_{a\ell} = \partial_a S^0_\ell$, in terms of the Schouten tensor associated to the connections Ξ_{MN} and Ξ^0_{MN}, respectively.

Proof. Due to the linearity of the equations (C.51), Σ_{MN} should satisfy

$$\partial_n \Sigma_{MN} = -\frac{1}{2}\partial_n(S_\ell - S^0_\ell) q_{MN}, \qquad D_{[M}\Sigma_{N]P} = -\frac{1}{2} D_{[M}(q_{N]P}(S_\ell - S^0_\ell)). \tag{C.53}$$

With the change of variables $\hat{\Sigma}_{MN} \equiv \Sigma_{MN} + \frac{1}{2}(S_\ell - S^0_\ell) q_{MN}$, the previous equations take the simpler form

$$\partial_n \hat{\Sigma}_{MN} = 0, \qquad D_{[M}\hat{\Sigma}_{N]P} = 0. \tag{C.54}$$

Contracting the second equation with q^{MP} we also find

$$q^{MP} D_{[M}\hat{\Sigma}_{N]P} = 0 \qquad \Longrightarrow \qquad D^N \hat{\Sigma}_{MN} = D_M \hat{\Sigma}^N_N. \tag{C.55}$$

In order to solve (C.54) and (C.55) consider the following decomposition of $\hat{\Sigma}_{MN}$ [78, 79]

$$\hat{\Sigma}_{MN} = D_{MN}\chi + D_{(M}A_{N)} + W_{MN} + t q_{MN}, \tag{C.56}$$

where χ and t are two scalar fields on $\mathcal{I}$ satisfying $\partial_n \chi = \partial_n t = 0$, and $t = \hat{\Sigma}^M_M$. The vector A_M and the tensor W_{MN} are both independent of the null coordinate $\partial_n A_M = \partial_n W_{MN} = 0$, and moreover W_{MN} is also transverse and traceless

$$D^N W_{MN} = 0, \quad \text{and} \quad W^M_M = 0. \tag{C.57}$$

The operator $D_{MN}\chi$ is defined by

$$D_{MN}\chi \equiv D_M D_N \chi - (\Delta + \frac{1}{2}\mathcal{R})\chi, \tag{C.58}$$

and has the property that $D_{MN}\chi$ is transverse for all scalar fields χ. As was proven in [78], in the previous decomposition the scalar χ can only be determined up to $\chi \to \chi + \lambda$ where λ is a solution to $(\Delta + \frac{1}{2}\mathcal{R})\lambda = 0$, and the vector A_M is fixed up to $A_M \to A_M + k_M$, where k_M is a killing vector of q_{MN}. The tensor W_{MN} and t are both uniquely determined by $\hat{\Sigma}_{MN}$. Inserting this decomposition (C.56) in the equation (C.55) we find

$$D^N D_{(M}A_{N)} = \partial_M t, \tag{C.59}$$

which can be solved by $A_M = \partial_M \phi$, where ϕ is a scalar satisfying $\Delta\phi + \frac{1}{2}\mathcal{R}\phi = \frac{1}{2}t$. The condition $\hat{\Sigma}^M_M = t$ leads to the equation

$$q^{MN}D_{MN}\chi - 2D^{M}A_{M} = 0 \qquad \Longrightarrow \qquad \Delta\chi + \mathcal{R}\chi = 2\Delta\phi. \tag{C.60}$$

The decomposition (C.56) is more conveniently expressed in terms of the combination $f \equiv \chi + 2\phi$, which satisfies $\Delta f + \mathcal{R}f = 2t$. We obtain the expression

$$\hat{\Sigma}_{MN} = D_M D_N f + \frac{1}{2}\mathcal{R}fq_{MN} + W_{MN}. \tag{C.61}$$

Substituting this result into (C.54) we find the following constraint for W_{MN}

$$D_{[M}W_{N]P} = 0. \tag{C.62}$$

Thus we can see the lemma C.1 applies to W_{MN}, since this tensor is constant along the null direction and traceless, and therefore we can conclude that $W_{MN} = 0$. Collecting these results, the final form of the solution to the equations (C.51) is

$$\Sigma_{MN} = D_M f_N + \frac{1}{2}\mathcal{R}fq_{MN} - \frac{1}{2}(S_\ell - S_\ell^0)q_{MN}, \tag{C.63}$$

which is the expression we were looking for. Note that the freedom to shift A_M by a killing vector leaves this expression invariant, while the ambiguity to shift $\chi \to \chi + \lambda$ amounts to shifting $f \to f + \lambda$. ■

It is trivial to check that (C.63) reduces to the form of the solution we presented in the main text, equation (5.26), when we express it in terms of the equivalence relation (5.17).

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Conclusions

CONCLUSIONS

This thesis is comprised of three articles and can be divided in two main parts. The first part, which consists on the first two articles, uses the holographic duality to study out of equilibrium physics in two different scenarios. The second part focuses on the asymptotic symmetries of non-expanding black hole horizons. We will briefly summarize the main results.

QUANTUM QUENCHES

We have proposed a simple holographic scenario, easily accessible to numerics, that models quantum quenches where a relevant coupling changes. In particular, we have constructed gravitational backgrounds dual to a family of quantum field theories parameterized by a relevant coupling. These backgrounds combine a non-trivial scalar field profile with a naked singularity. The naked singularity is necessary to preserve Lorentz invariance along the boundary directions. The singularity is however excised by introducing an infrared cutoff in the geometry, as has been done before to model confining theories. We have developed the holographic dictionary associated to the infrared boundary. We have implemented quenches between two different values of the coupling. This requires considering time dependent boundary conditions for the scalar field both at the AdS boundary and the infrared wall.

We have used radially localized gaussian pulses to represent the quench, and depending on its parameters such as the amplitude and time span, the endpoint of the evolution can be either the formation of a Schwarzschild black hole, or a bouncing geometry. The former is dual to a unitary process in the QFT leading to thermalization, while the latter represents periodic reconstructions of quantum correlations known as quantum revivals. We have analyzed this phenomenology and performed a number of checks.

ANOMALOUS TRANSPORT

In the presence of a gravitational contribution to the chiral anomaly, the chiral magnetic effect (CME) induces an energy current proportional to the square of the temperature in equilibrium. This energy current has two components, a non-dissipative one induced by the anomaly and a dissipative flow component. We have measured the dissipative component via the drag it asserts on an additional auxiliary color charge. Owing to momentum conservation, this provides us with a measure of the anomalous non-dissipative component.

In holography, the thermal state corresponds to a black hole. We have studied holographic quantum quenches of the CME induced by the gravitational anomaly. A rich phenomenology arises depending on the time scale of the quench. Intermediate quenches leave the far from equilibrium stage while they are reaching the final state, resulting in a monotonic growth of the current. Processes which are slow with respect to the final temperature but fast with respect to the initial one, have a finite period of near-equilibrium evolution. This extends to the complete evolution for large τT_0 (slower quenches). However, if the quench is fast with respect to the initial and final temperatures, the evolution is far from equilibrium until the final exponential approach to stabilization. The current builds up late in the time evolution and slightly overshoots before it achieves its equilibrium value.

The main motivation of this work was to analyze what activates the anomalous conductivity out of equilibrium, where there is no notion of temperature. It could have been governed by energy density, which in equilibrium is also measured by the temperature. In this case the current should have reacted as soon as energy is injected into the system. Our result on fast quenches shows that this is not the case. Rather the system has to evolve closer to equilibrium to build up the anomalous current. In other words, the energy current induced by the gravitational anomaly is very suppressed when the system is far from equilibrium.

HORIZON SUPERTRANSLATIONS

Recently it has been argued that the asymptotic symmetry group of spacetimes containing a non-extremal black hole should be enhanced with horizon supertranslations. These diffeomorphisms would transform the state of the black hole horizon in an analogous way as BMS supertranslations act on the geometric data of null infinity. According to this proposal, the multiplicity of black hole states generated by horizon supertranslations could provide a partial explanation for the Bekenstein-Hawking entropy formula.

In this work, we have characterized the geometrical nature of smooth supertranslations defined on a generic non-expanding horizon (NEH) embedded in vacuum. To this end, we have considered the constraints imposed by the vacuum Einstein's equations on the NEH structure, and discussed the transformation properties of their solutions under supertranslations. We have presented a freely specifiable data set

$$\text{Free horizon data:} \qquad \mathscr{D}_{free} \equiv (q_{MN}|_{\eta_0}, \quad \Omega^0_M|_{\eta_0}, \quad s_{MN}|_{\eta_0}).$$

which is both necessary and sufficient to reconstruct the full horizon geometry, and is composed of objects which are invariant under supertranslations. We concluded that smooth supertranslations do not transform the geometry of the NEH, and that they should be regarded as pure gauge. Our results apply both to stationary, and non-stationary states of a NEH, the latter ones being able to describe radiative processes taking place on the horizon.

As a consistency check we have repeated the analysis for BMS supertranslations defined on null infinity, $\mathcal{I}$. Using the same framework as for the NEH we recover the well known result that BMS supertranslations act non-trivially on the free data on $\mathcal{I}$.

CONCLUSIONES

Esta tesis está compuesta por tres artículos y se puede dividir en dos partes principales. La primera parte, que consiste en los dos primeros artículos, utiliza la dualidad holográfica para estudiar la física fuera del equilibrio en dos escenarios distintos. La segunda parte se centra en las simetrías asintóticas de agujeros negros que no se expanden. Vamos a resumir brevemente los principales resultados.

quenches cuánticos

Hemos propuesto un escenario holográfico simple, fácilmente accesible numéricamente, que modeliza quenches cuánticos donde un acoplo relevante varía. En particular, hemos construído backgrounds gravitacionales duales a una familia de teorías cuánticas de campos parametrizadas por un acoplo relevante. Estos backgrounds combinan un perfil no trivial de campo escalar y una singularidad desnuda. La singularidad desnuda es necesaria para preservar la invarianza Lorentz a lo largo de las direcciones de la frontera. Sin embargo, esta singularidad está escindida debido a la introducción de un corte (o *cutoff*) infrarrojo de la geometría, lo cual ya ha sido usado con anterioridad para modelar teorías con confinamiento. Hemos desarrollado el diccionario holográfico asociado a la frontera infrarroja. Hemos implementado quenches entre dos valores distintos del acoplo. Esto requiere considerar condiciones de frontera que dependen del tiempo para el campo escalar tanto en la frontera AdS como en la pared infrarroja.

Hemos usado pulsos gausianos radialmente localizados para representar el quench, y dependiendo de sus parámetros como la amplitud o la duración, el resultado final de la evolución puede ser un agujero negro de Schwarzschild o una geometría rebotante. El primer caso es dual a un proceso unitario en la QFT que resulta en la termalización del sistema, mientras que el segundo representa reconstrucciones periódicas de las correlaciones cuánticas que se conocen como *revivals* cuánticos. Hemos analizado esta fenomenología y realizado un número de comprobaciones.

transporte anómalo

En presencia de la contribución gravitacional a la anomalía quiral, el efecto magnético quiral (CME) induce una corriente de energía proporcional al cuadrado de la temperatura en equilibrio. Esta corriente de energía tiene dos componentes: una no-disipativa inducida por la anomalía y otra disipativa. Hemos medido la componente disipativa a través del rozamiento que hace sobre un campo auxiliar de color adicional. Debido a la conservación del momento, esto nos permite medir la componente anómala no disipativa.

En holografía, un estado termal se corresponde con un agujero negro. Hemos estudiado quenches cuánticos del CME inducidos por la anomalía gravitacional. Aparece una fenomenología muy rica dependiendo de la escala de tiempo del quench. Los quenches intermedios dejan el estado de fuera del equilibrio mientras se acercan al estado final, lo que resulta en un crecimiento monótono de la corriente. Los procesos que son lentos con respecto a la temperatura final, pero rápidos con respecto a la inicial, tienen un periodo finito de evolución cerca del equilibrio. Esto se extiende a una evolución completa para un valor grande de τT_0 (quenches más lentos). Sin embargo, si el quench es rápido con respecto a las temperaturas inicial y final, la evolución se produce fuera del equilibrio hasta la aproximación exponencial final al equilibrio. La corriente comienza a formarse bastante tarde en el proceso de evolución, y se pasa un poco del valor final antes de equilibrarse.

La principal razón de este trabajo era analizar qué es lo que activaba la conductividad anómala fuera del equilibrio, donde no existe una noción de la temperatura. La corriente podría haber estado gobernada por la densidad de energía, que en equilibrio también se mide por la temperatura. En este caso, la corriente debería haber reaccionado tan pronto como se le inyectaba energía al sistema. Nuestro resultado en quenches rápidos muestra que este no es el caso. Más bien el sistema tiene que evolucionar hasta cerca del equilibrio para formar la corriente anómala. En otras palabras, la corriente de energía inducida por la anomalía gravitacional está muy suprimida cuando el sistema se encuentra lejos del equilibrio.

SUPERTRASLACIONES EN EL HORIZONTE

Recientemente se ha discutido la posiblidad de que el grupo de simetrías asintóticas de espacio-tiempos que contienen un agujero negro no-extremal se pueda aumentar con supertraslaciones en el horizonte. Estos difeomorfismos transformarían el estado del horizonte del agujero negro de una forma análoga a como las supertraslaciones BMS actúan sobre los datos geométricos en el infinito nulo. Según esta propuesta, la multiplicidad de estados de agujero negro generada por supertraslaciones del horizonte podría proporcionar una explicación parcial para la fórmula de la entropía de Bekenstein y Hawking.

En este trabajo, hemos caracterizado la naturaleza geométrica de supertraslaciones sin singularidades definidas en un horizonte genérico que no expande (NEH) embebido en el vacío. Para ello, hemos considerado las restricciones impuestas por las ecuaciones de Einstein de vacío en la estructura del NEH, y hemos discutido las propiedades de transformación de sus soluciones bajo supertraslaciones. Hemos presentado un set de datos que se pueden especificar libremente

$$\text{Datos libres del horizonte:} \qquad \mathscr{D}_{free} \equiv (q_{MN}|_{\eta_0}, \quad \Omega^0_M|_{\eta_0}, \quad s_{MN}|_{\eta_0}).$$

que es necesario y suficiente para reconstruir la geometría completa del horizonte, y está compuesto de objetos que son invariantes bajo supertraslaciones. Hemos concluido que las supertraslaciones sin singularidades (suaves) no transforman la geometría del NEH, y que deben ser consideradas como gauge puro. Nuestros resultados aplican tanto a estados estacionarios y no estacionarios del NEH, donde estos últimos pueden describir procesos radiativos ocurriendo en el horizonte. Como comprobación de consistencia hemos repetido este análisis para supertraslaciones BMS definidas en el infinito nulo $\mathcal{I}$. Usando la misma formulación que para el NEH hemos recuperado el conocido resultado de que las supertraslaciones BMS actúan no trivialmente en los datos libres en $\mathcal{I}$.

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